Worksheet

1/10/2020

1. Approximate the value of e with acceptable error $\epsilon = 10^{-3}$ by using Taylor approximation for the function $f(x) = e^x$ about $x_0 = 0$.

Note that $e = f(1)$. We have

$$f(x) = f'(x) = f''(x) = f^{(3)}(x) = \dots = e^x$$

and $f(0) = f'(0) = f''(0) = \dots = 1$.

The n'th Taylor polynomial of f is

$$p_n(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \dots + \frac{f^{(n)}(0)}{n!}(x-0)^n$$

$$= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n.$$

We have

$$f(x) = p_n(x) + \underbrace{R_n(x)}_{\text{remainder}}.$$

At $x = 1$,

$$f(1) = p_n(1) + R_n(1).$$

By Lagrange formula,

$$R_n(1) = \frac{f^{(n+1)}(c)}{(n+1)!}(1-0)^{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!} = \frac{e^c}{(n+1)!}$$

where c is some number between $x_0 = 0$ and $x = 1$.

Then

$$|R_n(1)| = \frac{e^c}{(n+1)!} < \frac{e}{(n+1)!} < \frac{3}{(n+1)!}$$

To make sure that $|R_n(1)| < 10^{-3}$, we need to choose n sufficiently large such that

$$\frac{3}{(n+1)!} < 10^{-3}.$$

Using calculator, we can choose $n = 6$. Then

$$e = f(1) \approx P_6(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{6!}$$

$$\approx 2.7181 \quad \text{(using calculator)}$$

One can also write a Matlab program to compute this sum.

in a script file (.m)

```
s = 1 ;
for k=1:6
      s = s + 1/factorial(k);
end
s
```

2. How large should n be so that the function $f(x) = e^{-2x}$ can be approximated by its n'th Taylor polynomial $p_n(x)$ within error tolerance $\epsilon = 10^{-6}$ for all $x \in (-2, 1)$?

We already know that

$$g(t) = e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!} + R_n(t)$$

Replace t by $-2x$:

$$\underbrace{e^{-2x}}_{f(x)} = \underbrace{1 + \frac{-2x}{1!} + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \cdots + \frac{(-2x)^n}{n!}}_{\substack{q_n(x) \\ \text{(the n'th Taylor polg. of f)}}} + \underbrace{R_n(-2x)}_{\substack{\tilde{R}_n(x) \\ \text{(the remainder)}}}$$

We want to find n such that $|\tilde{R}(x)| = |R_n(-2x)| < 10^{-6}$.

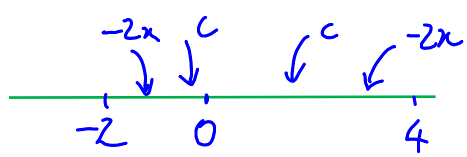
Put $t = -2x$.

Apply Lagrange's theorem for function g:

$$R_n(t) = \frac{g^{(n+1)}(c)}{(n+1)!} t^{n+1} = \frac{e^c}{(n+1)!} t^{n+1}$$

where c is some number between 0 and $t = -2x$.

When x varies on $(-2, 1)$, $t = -2x$ varies on $(-2, 4)$.



We see that c must lie between -2 and 4. Thus,

$$|R_n(-2x)| = \frac{e^c}{(n+1)!}|-2x|^{n+1} \leq \frac{e^4 4^{n+1}}{(n+1)!} < \frac{3^4 4^{n+1}}{(n+1)!} = \frac{81 \cdot 4^{n+1}}{(n+1)!}$$

Therefore, the error term $\tilde{R}_n(x)$ is bounded by

$$|\tilde{R}_n(x)| < \frac{81 \cdot 4^{n+1}}{(n+1)!} \qquad \forall\, x \in (-2, 1).$$

To make sure that $|\tilde{R}_n(x)| < 10^{-6}$ for all $x \in (-2, 1)$, we only need to choose n such that

$$\frac{81 \cdot 4^{n+1}}{(n+1)!} < 10^{-6}.$$

$n = 20$ will do it (checking by calculator).