MATH 351, MIDTERM EXAM, WINTER 2020

Name	Student ID

- Write your solution to each problem in a readable manner. Circle your final results.
- Show all your work. Answers not supported by valid arguments will get little or no credit. You can use the blank page on the back of the exam if you need more space.
- Doing correctly Problems 1, 2, 3, 4, 5 will grant you 100% credit of the exam. You can earn extra credit by doing Problem 6.

Problem	Possible points	Earned points
1	10	
2	10	
3	10	
4	10	
5	10	
6	5	
Total	55	

Some formula:

$$n \geq \log_2\left(\frac{b-a}{\epsilon}\right) - 1,$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$|x_{n+1} - \alpha| \leq C|x_n - \alpha|^p.$$

Problem 1. (10 points) How big should n be so that e can be approximated by $1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$ with error less than 0.001?

How big should n be so that

$$e \approx 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

with error $< 10^{-3}$?

Consider function $f(x) = e^x$. The n'th Taylor expansion of f is

$$f(x) = e^x = \underbrace{1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}_{p_n(x)} + R_n(x)$$

For $x = 1$:

$$e = \underbrace{1 + \frac{1}{1!} + \cdots + \frac{1}{n!}}_{p_1(1)} + R_n(1)$$

The error term is $R_n(1)$ By Lagrange theorem,

$$R_n(1) = \frac{f^{(n+1)}(c)}{(n+1)!}(1-0)^{n+1} = \frac{e^c}{(n+1)!}$$

for some c between 0 and 1

Then $0 < R_n(1) \leq \frac{e}{(n+1)!} < \frac{3}{(n+1)!}$

To make sure that $R_n(1) < 10^{-3}$, we choose large n such that

$$\frac{3}{(n+1)!} < 10^{-3}.$$

By calculator, we see that $n \geq 6$ would do it.

Consider the following toy model of the IEEE double precision floating-point format:
The sequence of 8 bits

$$\underbrace{c_0}_{\text{sign part}} \quad \underbrace{b_1 \quad b_2 \quad b_3 \quad b_4}_{\text{exponent part}} \quad \underbrace{a_1 \quad a_2 \quad a_3}_{\text{mantissa part}}$$

represents a number $x = \sigma \cdot \bar{x} \cdot 2^e$ where σ, $\bar{x}$, e are determined as follows:

$$\begin{aligned} \sigma &= \begin{cases} 1 & \text{if } \; c_0 = 0, \\ -1 & \text{if } \; c_0 = 1, \end{cases} \\ E &= (b_1 b_2 b_3 b_4)_2 \end{aligned}$$

- If $1 \le E \le 14$ then

$$\begin{aligned} e &= E - 7, \\ \bar{x} &= (1.a_1 a_2 a_3)_2 \end{aligned}$$

- If $E = 0$ then $e = -6$ and $\bar{x} = (0.a_1 a_2 a_3)_2$.
- If $E = 15$ then $x = \pm\infty$ (depending on the sign σ).

Problem 2. (10 points) Write number 3.7 in this floating-point system (in form of $\sigma \cdot \bar{x} \cdot 2^e$).

$3.7 = 3 + 0.7$

$3 = (11)_2$

Convert 0.7 into binary:

$0.7 \times 2 = 1.4 \longrightarrow 1$

$0.4 \times 2 = 0.8 \longrightarrow 0$

$0.8 \times 2 = 1.6 \longrightarrow 1$

$0.6 \times 2 = 1.2 \longrightarrow 1$

$0.2 \times 2 = 0.4 \longrightarrow 0$

$0.4 \times 2 = 0.8 \longrightarrow 0$

repeated pattern.

Thus, $0.7 = (0.1\underbrace{0110}\,\underbrace{0110}\ldots)_2$

Then $3.7 = (11.1\underbrace{0110}\,\underbrace{0110}\ldots)_2$

$= (1.11011001100110\ldots)_2 \times 2^1$

$\approx (1.111)_2 \times 2^1.$

Problem 3. (10 points) Let $x = (1.101)_2 \times 2^2$ and $y = (1.011)_2 \times 2^3$. Perform the multiplication $x \cdot y$ in the floating-point system given on the previous page.

$$x \cdot y = (1.101)_2 \times (1.011)_2 \times 2^5$$

Multiply the significands:

$$\begin{array}{r} 1.101 \\ \times\ 1.011 \\ \hline 1101 \\ 1101 \\ 0000 \\ 1101 \\ \hline 10.001111 \end{array}$$

Then

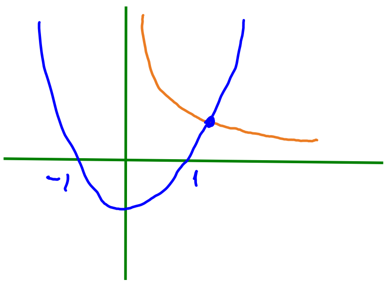
$$\begin{aligned} x \cdot y &= (10.001111)_2 \times 2^5 \\ &= (1.0001111)_2 \times 2^6 \\ &\approx (1.001)_2 \times 2^6 \end{aligned}$$

Problem 4. (10 points)

(a) Sketch the graphs of $f(x) = \frac{1}{x}$ and $g(x) = x^2 - 1$.

(b) Use a suitable numerical method to find an approximate value of the x-coordinate of the intersection point of the two graphs on the half-plane $x > 0$. The allowed error is 0.1.

.



The intersection point solves the equation

$$\frac{1}{x} = x^2 - 1,$$

which is equivalent to $x^3 - x - 1 = 0$.

Put $h(x) = x^3 - x - 1$.

The picture gives us a hint that h as only one positive root and this root is larger than 1. We have

$$h(1) = -1 < 0, \qquad h(2) = 5 > 0.$$

Thus, h as a root on the interval $[1, 2]$.

We will use Bisection method with the initial interval $[a_0, b_0] = [1, 2]$.

The number of steps that need to be done is

$$n \geq \log_2 \frac{b_0 - a_0}{2} - 1 = \log_2 \frac{2-1}{0.1} - 1 \approx 2.32$$

Hence, $n = 6$ steps would be sufficient

$a_0 = 1, \quad b_0 = 2, \quad c_0 = 1.5, \quad f(c_0) = 0.875 > 0.$

$a_1 = 1, \quad b_1 = c_0 = 1.5, \quad c_1 = 1.25, \quad f(c_1) = -0.2968\ldots < 0$

$a_2 = c_1 = 1.25, \quad b_2 = b_1 = 1.5, \quad c_2 = 1.375, \quad f(c_2) = 0.22\ldots > 0$

$a_3 = a_2 = 1.25, \quad b_3 = c_2 = 1.375, \quad c_3 = 1.3125$

Problem 5. (10 points) Consider a sequence defined recursively as

$$x_{n+1} = \frac{x_n^3 - 2x_n^2 + 10}{5}, \qquad x_0 = 1.$$

(1) Use your pocket calculator to guess the limit of this sequence. Then use the recursive formula to verify that this number is truly a limit of (x_n).
(2) Find the order of convergence. If the order of convergence is 1, find the linear rate of convergence.

$x_1 = 1.8$, $x_2 = 1.8764$, $x_3 = 1.909$, $x_4 = 1.9338$, $x_5 = 1.95 \ldots$

We guess that the limit is 2.

To check if 2 is truly a limit of (x_n), we put $a = 2$. Take the limit of both sides:

$$a = \frac{a^3 - 2a^2 + 10}{5}.$$

This is equivalent to $a^3 - 2a^2 - 5a + 10 = 0$.

Factor $a-2$. $(a-2)(a^2-5) = 0$

This gives $a = 2$ and $a = \pm\sqrt{5}$.

Because the sequence is close to 2, the limit must be $a = 2$.

Find order of convergence.

$$x_{n+1} - 2 = \frac{x_n^3 - 2x_n^2 + 10}{5} - 2 = \frac{x_n^3 - 2x_n^2}{5} = \frac{x_n^2(x_n - 2)}{5}.$$

Take the absolute value of both sides.

$$e_{n+1} = \frac{x_n^2}{5} e_n \approx \frac{4}{5} e_n \quad \text{(when } n \text{ is large)}$$

Thus, the order of convergence is 1 and the linear rate of convergence is $4/5$.

Problem 6. (5 points) Give an example of a function f and a real number x_0 such that the iteration formula of the Newton's method, starting with x_0, gives a divergent sequence. Explain your example.

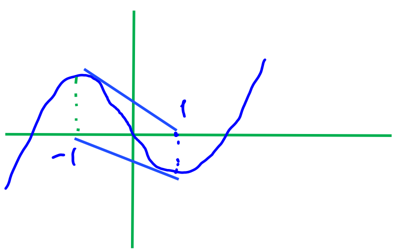
$$f(x) = x^3 - 5x \quad , \quad x_0 = 1$$

$$f'(x) = 3x^2 - 5$$

$$x_{n+1} = x_n - \frac{x_n^3 - 5x_n}{3x_n^2 - 5} = \frac{2x_n^3}{3x_n^2 - 5}$$

We get $x_1 = -1$, $x_2 = 1$, $x_3 = -1$, $x_4 = 1, \ldots$

This sequence diverges.



Note: there are a lot of other examples. Start with a function f that has no roots. For example $f(x) = x^2 + 1$. Then x_n will diverge

$$x_{n+1} = x_n - \frac{x_n^2 + 1}{2x_n} = \frac{x_n^2 - 1}{2x_n} = \frac{x_n}{2} - \frac{1}{2x_n}.$$

With $x_0 = 2$, we get $x_1 = \frac{3}{4}$, $x_2 = -0.29$, $x_3 = 1.568\ldots$

$x_4 = 0.465, \ldots$