Lecture 3

Friday, January 10, 2020

Continue the example last time:

$$\sqrt{3} = f(3) = p_n(3) + R_n(3),$$

where

$$p_n(3) = f(4) + \frac{f'(4)}{1!}(-1) + \frac{f''(4)}{2!}(-1)^2 + \cdots + \frac{f^{(n)}(4)}{n!}(-1)^n,$$

$$f^{(k)}(4) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(k-1)\right) 2^{1-2k}.$$

We see that given any n, one can compute $p_n(3)$ by hand. The only task we have to do is to make sure that error term is bounded by 10^{-6}. That is $|R_n(3)| < 10^{-6}$ Note that n is the only degree of freedom we can choose.

At this step, Lagrange's theorem become useful Lagrange says that the error term $R_n(3)$ has the form

$$R_n(3) = \frac{f^{(n+1)}(c)}{(n+1)!}(3-4)^{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!}(-1)^{n+1}$$

where c is some number between $x=4$ and $x_0=3$.

Note that we are not interested in the sign of $R_n(3)$, only in the size (magnitude) of its. Take the absolute value of both sides:

$$|R_n(3)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}$$

Recall the following properties of absolute values:

- $|ab| = |a|\,|b|$
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
- $|a^n| = |a.a\ldots a| = |a||a|\ldots|a| = |a|^n$
- $|a+b| \le |a|+|b|$ (triangle inequality)
- $|a+b-c| \le |a|+|b|+|-c| = |a|+|b|+|c|$

These are useful for us to estimate the error term. We want to see how big $R_n(3)$ can be in terms of n:

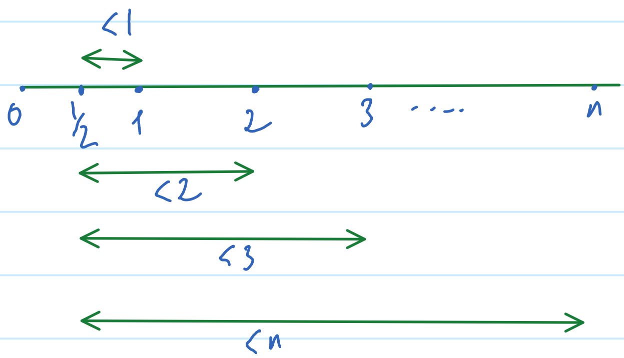
We derived that $f^{(k)}(x) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(k-1)\right) x^{\frac{1}{2}-n}$

Thus, $f^{(n+1)}(c) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n\right) c^{-\frac{1}{2}-n}$.

Take the absolute value of both sides.

$$|f^{(n+1)}(c)| = \frac{1}{2}\left|\frac{1}{2}-1\right|\left|\frac{1}{2}-2\right|\cdots\left|\frac{1}{2}-n\right| c^{-\frac{1}{2}-n}$$

Because c is somewhere between 3 and 4, we have $c^{-\frac{1}{2}-n} \leq 3^{-\frac{1}{2}-n}$.



$\left|\frac{1}{2}-1\right| < 1$

$\left|\frac{1}{2}-2\right| < 2$

....

$\left|\frac{1}{2}-n\right| < n$

Thus, $|f^{(n+1)}(c)| < \frac{1}{2}\cdot 1\cdot 2 \cdots n \cdot 3^{-\frac{1}{2}-n} = \frac{1}{2} n!\, 3^{-\frac{1}{2}-n}$

Plug this into the formula of $R_n(3)$:

$$|R_n(3)| < \frac{\frac{1}{2} n!\, 3^{-\frac{1}{2}-n}}{(n+1)!} = \frac{1}{2}\frac{1}{n+1} 3^{-\frac{1}{2}-n}.$$

In order to have $|R_n(3)| < 10^{-6}$, we only need to choose n such that

$$\frac{1}{2}\frac{1}{n+1} 3^{-\frac{1}{2}-n} < 10^{-6}.$$

Any $n \geq 10$ would do it (by testing with calculator). However, we are interested in small n (say $n=10$) because it takes less computation to find $p_n(3)$.

[see more practice on the worksheet]