Lecture 27

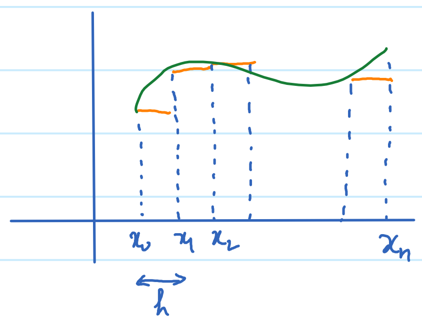
Wednesday, March 11, 2020

Last time we obtained an error estimate for the left-point Riemann sum. Specifically, we showed that

$$|L_n - I| \le \frac{(b-a)^2}{2n} \max_{[a,b]} |f'|$$

The key step to obtain this estimate is using 0'th Taylor approximation

$$f(x) = f(x_0) + \frac{f'(c_x)}{1!}(x - x_0) \qquad \forall x \in [x_0, x_1]$$



The error coming from the approximation on the first interval

$$\int_{x_0}^{x_1} f(x)\,dx \approx h\, f(x_0)$$

is at most $\displaystyle \frac{(b-a)^2}{2n^2} \max |f'|$ $\quad (*)$

Because the approximation $I \approx L_n$ is constituted by n of such approximations (one on each subinterval), the total error is at most n times the error $(*)$, which is

$$\frac{(b-a)^2}{2n} \max_{[a,b]} |f'|$$

For the right-hand point, we use similar Taylor approximation on $[x_0, x_1]$ except that the base point is x_1:

$$f(x) = f(x_1) + \frac{f'(d_x)}{1!}(x - x_1)$$

We obtain the same error bound:

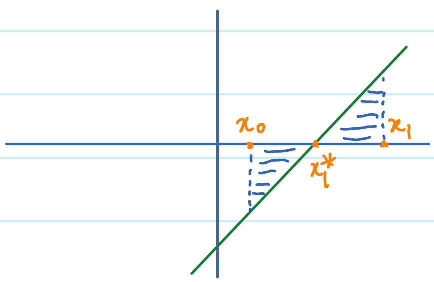
$$|R_n - I| \le \frac{(b-a)^2}{2n} \max_{[a,b]} |f'|$$

For midpoint rule, one can use the same approach, with the base point of

Taylor approximation being the midpoint $x_1^* = \frac{x_0 + x_1}{2}$, to get the same error bound. However, this doesn't explain why the midpoint rule is better than the left/right-point rule as one observes on pictures or experiments. The only adjustment in the way we bound the error is to use first Taylor approximation instead of zeroth Taylor approximation.

$$f(x) = f(x_1^*) + f'(x_1^*)(x - x_1^*) + \frac{f''(c_x)}{2!}(x - x_1^*)^2 .$$

Note that when we take the integral of both sides over $[x_0, x_1]$, the second term on RHS vanishes away. This is because



$$\int_{x_0}^{x_1} (x - x_1^*)\, dx = 0 .$$

With this Taylor approximation, one can show that $\int_{x_0}^{x_1} f(x)\, dx \approx h f(x_1^*)$

with error bounded by $\frac{h^3}{24} \max |f''|$.

We then get $$|M_n - I| \le \frac{(b-a)^3}{24 n^2} \max_{[a,b]} |f''| .$$

For trapezoid rule, one can use first Taylor approximation about x_0 to obtain $$|T_n - I| \le \frac{(b-a)^3}{12 n^2} \max_{[a,b]} |f''| .$$

We see that the error of the midpoint rule and trapezoid rule go to 0 at the rate of $1/n^2$, whereas the error of the left-point and right-point rule go to 0 at the rate of $1/n$. With essentially the same amount of calculation, the first two rules are therefore more efficient.

[See examples on worksheet.]