Lecture 26

Monday, March 9, 2020

Last time, we discussed some situations where numerical integration is necessary. For example, the function $f(x) = \frac{\sin x}{x}$ doesn't have an antiderivative that is an elementary function. However, the antiderivative can be expressed as a power series.

$$\frac{\sin x}{x} = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

An antiderivative of this function is

$$x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \cdots$$

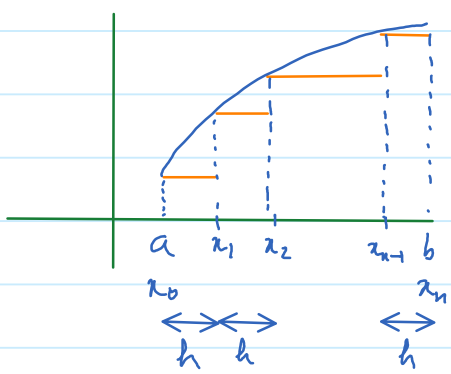
Thus,

$$\int_1^2 \frac{\sin x}{x}\,dx = \left(x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \cdots\right)\Bigg|_1^2$$

$$= \left(2 - \frac{2^3}{3\cdot 3!} + \frac{2^5}{5\cdot 5!} - \cdots\right) - \left(1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \cdots\right)$$

Riemann sums are in general easy to compute. All we need to evaluate approximately $\int_a^b f(x)\,dx$ is to evaluate f at sample points $x_0, x_1, \ldots, x_n$ on the interval $[a,b]$. For example,

$$\underbrace{\int_a^b f(x)\,dx}_{I} \approx \underbrace{h f(x_0) + h f(x_1) + \cdots + h f(x_{n-1})}_{L_n}$$



We are interested in how large the error $|I - L_n|$ can be with respect to n. The number n is called a control parameter because we are free to choose it. The number L_n is called an observable because it is only known after an experiment.

One can see that the approximation $L_n \approx I$ is constituted by the many approximations at smaller scale:

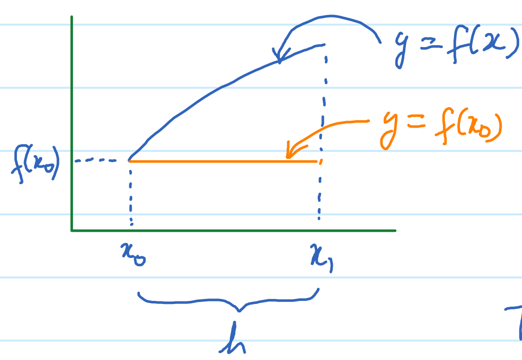
$$I = \underbrace{\int_a^b f(x)\,dx}_{\text{total area}} = \underbrace{\int_{x_0}^{x_1} f(x)\,dx}_{\substack{\text{area above}\\ [x_0,x_1]}} + \underbrace{\int_{x_1}^{x_2} f(x)\,dx}_{\substack{\text{area above}\\ [x_1,x_2]}} + \cdots + \underbrace{\int_{x_{n-1}}^{x_n} f(x)\,dx}_{\substack{\text{area above}\\ [x_{n-1},x_n]}}.$$

The area above each subinterval is approximated by the area of a rectangle.

$$L_n = \underbrace{hf(x_0)}_{\substack{\text{rectangle}\\ \text{above } [x_0,x_1]}} + \underbrace{hf(x_1)}_{\substack{\text{rectangle}\\ \text{above } [x_1,x_2]}} + \cdots + \underbrace{hf(x_{n-1})}_{\substack{\text{rectangle}\\ \text{above } [x_{n-1},x_n]}}.$$

To estimate $|I - L_n|$, we only need to estimate the error at each subinterval $[x_k, x_{k+1}]$. Then the total error $|I - L_n|$ is just an accumulation of those errors.

Let us consider the first interval $[x_0, x_1]$. The area of the rectangle is the area under the constant function $y = f(x_0)$. One can write



$$h f(x_0) = \int_{x_0}^{x_1} f(x_0)\,dx.$$

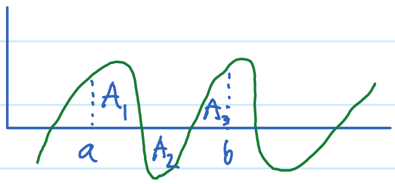
The difference between the true area and the approximated area is

$$\int_{x_0}^{x_1} f(x)\,dx - hf(x_0) = \int_{x_0}^{x_1} f(x)\,dx - \int_{x_0}^{x_1} f(x_0)\,dx = \int_{x_0}^{x_1} (f(x) - f(x_0))\,dx.$$

Take the absolute value of both sides:

$$\left| \int_{x_0}^{x_1} f(x)\,dx - hf(x_0) \right| = \left| \int_{x_0}^{x_1} (f(x) - f(x_0))\,dx \right| \le \int_{x_0}^{x_1} |f(x) - f(x_0)|\,dx. \quad (*)$$

Here we have just used the following general property of integral:



$$\left|\int_a^b g(x)\,dx\right| \le \int_a^b |g(x)|\,dx. \qquad (**)$$

As illustrated on the picture,

$$\int_a^b g(x)\,dx = A_1 - A_2 + A_3 \quad \text{and}$$

$$\int_a^b |g(x)|\,dx = A_1 + A_2 + A_3.$$

By triangle inequality, $|A_1 - A_2 + A_3| \le A_1 + A_2 + A_3$. This explains why we have $(**)$.

Return to the problem: our task is to see how big $|f(x) - f(x_0)|$ can be. Here we are looking for the error coming from approximating f by a constant function $y = f(x_0)$. This is zero'th Taylor approximation.

$$f(x) = \underbrace{p_0(x)}_{=f(x_0)} + R_0(x)$$

By Lagrange theorem,
$$R_0(x) = \frac{f'(c_x)}{1!}(x - x_0)$$

for some c_x between x_0 and x. Note that $x \ge x_0$ since x belongs to the interval $[x_0, x_1]$. Therefore,

$$|f(x) - f(x_0)| = |f'(c_x)|\,(x - x_0) \le K\,(x - x_0)$$

where $K = \max\limits_{[a,b]} |f'|$. Note that K does not depend on x.

Substituting this estimate into $(*)$, we get

$$\left|\int_{x_0}^{x_1} f(x)\,dx - hf(x_0)\right| \le \int_{x_0}^{x_1} K(x - x_0)\,dx = K\frac{(x_1 - x_0)^2}{2} = \frac{(b-a)^2}{2n^2}K.$$

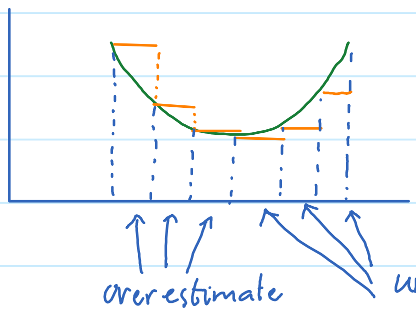
We have showed that the error between the exact area on the interval $[x_0, x_1]$ and the area of the rectangle is at most $\delta = \dfrac{(b-a)^2}{2n^2}K$.

On each of the other subintervals, the error is also at most δ.

Therefore, the total error $|L_n - I|$ is at most

$$\underbrace{\delta + \delta + \cdots + \delta}_{n \text{ times}} = n\delta = \frac{(b-a)^2}{2n} K .$$

This is the worst error possible. The actual error may be far less than this. For example, if f is not increasing nor decreasing as in the picture then the errors at different subintervals may cancel each other.



* **Theorem:**

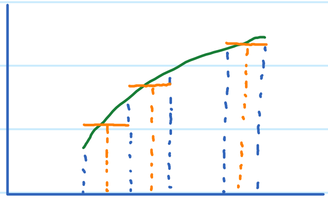
$$|L_n - I|, \ |R_n - I| \le \frac{(b-a)^2}{2n} K,$$

$$|M_n - I| \le \frac{(b-a)^3}{24n^2} \tilde{K},$$

$$|T_n - I| \le \frac{(b-a)^3}{12n^2} \tilde{K},$$

where $K = \max_{[a,b]} |f'|$ and $\tilde{K} = \max_{[a,b]} |f''|$.

We see that the midpoint method is better error than the left and right point method. This is because there are cancelations of error even in each subinterval.



To compute L_n, we need to evaluate f at n sample points $x_0, x_1, \ldots, x_{n-1}$.

To compute R_n, we need to evaluate f at n sample points $x_1, x_2, \ldots, x_n$.

To compute M_n, we need to evaluate f at n sample points:

$$x_1^* = \frac{x_0 + x_1}{2}, \quad x_2^* = \frac{x_1 + x_2}{2}, \ \ldots, \ x_{n-1}^* = \frac{x_{n-1} + x_n}{2}.$$

We see that with essentially the same amount of calculation, the midpoint method gives a better approximation. Therefore, the midpoint method is more preferred than the left-point or right-point method.