Lecture 25

Friday, March 6, 2020

We have finished an important topic in Numerical Analysis, namely Interpolation. We will now discuss another classic topic of Numerical Analysis, namely Numerical Integral. Let us consider a few motivating examples.

Ex :
$$\int_0^1 \sin x \, dx$$

One should have no trouble finding this integral since $\sin x$ has a nice antiderivative. By the Fundamental theorem of Calculus,
$$\int_0^1 \sin x \, dx = -\cos x \Big|_0^1 = 1 - \cos 1.$$

Ex :
$$\int_0^1 x e^x \, dx$$

In Calculus I, one is probably familiar with this integral. It is computed using integration by part:

$$du = e^x dx \qquad \rightsquigarrow \qquad u = e^x$$
$$v = x \qquad \qquad \qquad dv = dx$$

Ex
$$\int_0^1 x^{93} e^x \, dx$$

One can find an antiderivative of $x^{93} e^x$ by using integration by part 93 times. Then the integral will be equal to a sum of many numbers, including e and many fractions. It still takes considerable computation effort to go from here to a numerical answer.

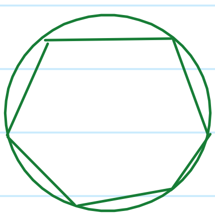
Ex
$$\int_0^1 e^{e^x} dx$$

The antiderivative of e^{e^x} is not an elementary function (i.e. finite combination of basic functions such as polynomials, exponent functions, trigonometric functions, etc). Thus, the Fundamental Theorem of Calculus is not effective.

Numerical integrations are techniques to find an approximate numerical value of an integral without requiring antiderivative of the integrand. To be more specific, let us formula the problem as follows.

Given a continuous function $f: [a,b] \to \mathbb{R}$ and a number $\varepsilon > 0$. Find an approximate numerical value of the integral $\int_a^b f(x)\,dx$ with error not exceeding ε.

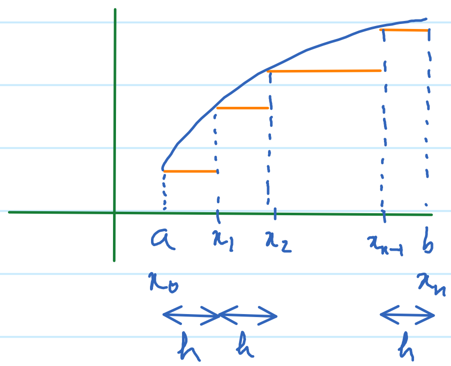
In many applications, this is all we need. Numerical integration is a very old problem: people had used numerical method to compute the area of different shape long before they discovered more sophisticated analytic theories such as the Fundamental Theorem of Calculus. For example, Archemedes (~200BC) used regular polygons to approximate a circle, thereby approximating the area of a circle.



In 1868, Riemann used a similar technique to approximate the area under any continuous curve. Another way to approximate $\int_a^b f(x)\,dx$ is to approximate f by a polynomial or a piece-wise polynomial curve (spline). Then $\int_a^b f(x)\,dx$ will be approximated by the integral of this curve (which is computable).

Let us first consider Riemann's technique. The integral $\int_a^b f(x)$ is approximated by Riemann sums.

There are different types of Riemann sums : left-point, right-point, midpoint.



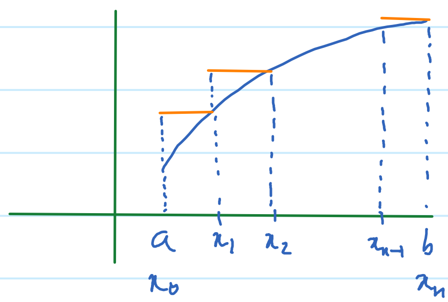
Let us partion the interval $[a,b]$ into equal subintervals $[x_0,x_1], [x_1,x_2], \dots, [x_{n-1},x_n]$.
We "sample" f at the points $x_0, x_1, \dots, x_n$.
That is to evaluate f at these points.

* Left-point Riemann sum :

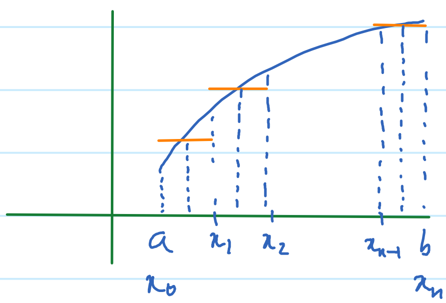
$$\int_a^b f(x)\,dx \approx h f(x_0) + h f(x_1) + \dots + h f(x_{n-1}) = \underbrace{h \sum_{k=0}^{n-1} f(x_k)}_{L_n} .$$

* Right-point Riemann sum:

$$\int_a^b f(x)\,dx \approx h f(x_1) + h f(x_2) + \dots + h f(x_n) = \underbrace{h \sum_{k=1}^{n} f(x_k)}_{R_n}$$



* Midpoint Riemann sum :

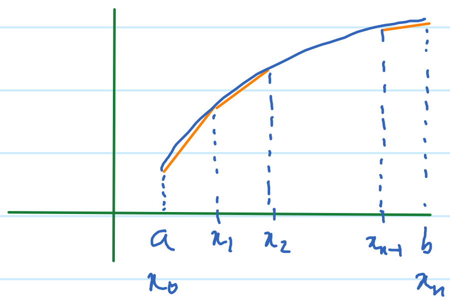


$$\int_a^b f(x)\,dx \approx h f\left(\frac{x_0+x_1}{2}\right) + \dots + h f\left(\frac{x_{n-1}+x_n}{2}\right)$$

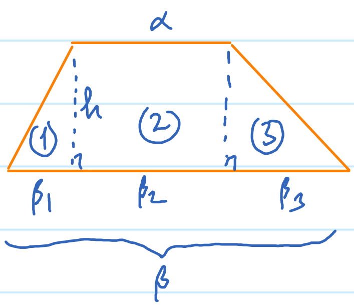
$$= \underbrace{h \sum_{k=0}^{n-1} f\left(\frac{x_k+x_{k+1}}{2}\right)}_{M_n}$$

$*$ Trapezoid sum:

Trapezoid sum is different from the previous sums in that we don't approximate the curve $y=f(x)$ by a step function ("zero'th order" spline curve), but rather by a linear spline curve.



The area of each slat under the true curve and above $[x_0, x_1]$ is approximated by the area of the trapezoid as indicated in the picture.



Recall how to evaluate the area of the trapezoid whose base lengths are α and β, and height is h. In the picture, we have

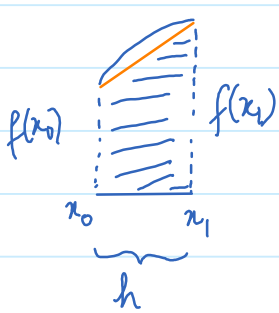
$$\text{Area} = ① + ② + ③$$

$$= \frac{1}{2}\beta_1 h + \beta_2 h + \frac{1}{2}\beta_3 h$$

$$= \frac{1}{2}(\beta_1 + 2\beta_2 + \beta_3)h$$

$$= \frac{1}{2}(\beta + \beta_2)h = \frac{1}{2}(\beta + \alpha)h.$$

In other words, the area of a trapezoid is equal to the product of the height and the average of the bases. Return to the problem, the area of the first trapezoid is $\frac{1}{2}h\left(f(x_0) + f(x_1)\right)$.



Therefore,

$$\int_a^b f(x)\,dx \approx \frac{1}{2}h\left(f(x_0)+f(x_1)\right) + \dots + \frac{1}{2}h\left(f(x_{n-1})+f(x_n)\right)$$

$$= \underbrace{\frac{1}{2}h\left(f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n)\right)}_{T_n}$$