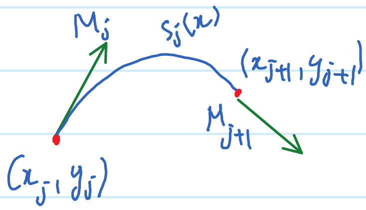
Lecture 24

Wednesday, March 4, 2020

Last time we learned how to find a quadratic spline that fits $n=3$ given points. Let us consider how to generalize our method for a general n & how to plot the quadratic spline curve on Matlab.



Given n data points $(x_1, y_1), \dots, (x_n, y_n)$, we want to find $n-1$ parabolae

$$s_1(x) = a_1 x^2 + b_1 x + c_1,$$
$$s_2(x) = a_2 x^2 + b_2 x + c_2,$$
$$\dots\dots$$
$$s_{n-1}(x) = a_{n-1} x^2 + b_{n-1} x + c_{n-1}.$$

Here $s_j(x)$ is a parabola joining two points (x_j, y_j) and (x_{j+1}, y_{j+1}). We need to find a_j, b_j, c_j for $j=1,2,\dots,n-1$. These are "local" parameters because they are parameters of local curves $s_1, s_2, \dots, s_{n-1}$. Our strategy is to determine them through the "global" parameters $M_1, M_2, \dots, M_n$, where M_j is the slope of the spline curve at point (x_j, y_j).

The three conditions

$$s_j'(x_j) = M_j,$$
$$s_j'(x_{j+1}) = M_{j+1},$$
$$s_j(x_j) = y_j$$

help us determine a_j, b_j, c_j in terms of M_j and M_{j+1}. Specifically,

$$a_j = \frac{M_j - M_{j+1}}{2(x_j - x_{j+1})},$$

$$b_j = \frac{x_j M_{j+1} - x_{j+1} M_j}{x_j - x_{j+1}},$$

$$c_j = y_j - a_j x_j^2 - b_j.$$

[Detail calculation is found in Lecture 23.]

The condition the parabola must pass through the right point (x_{j+1}, y_{j+1})

gives us an equation which M_j and M_{j+1} must satisfy. That is

$$a_j x_{j+1}^2 + b_j x_{j+1} + c_j = y_{j+1}$$

or equivalently

$$a_j (x_{j+1}^2 - x_j^2) + b_j (x_{j+1} - x_j) = y_{j+1} - y_j .$$

[We have used the fact that $c_j = y_j - a_j x_j^2 - b_j x_j$.]

After plugging a_j and b_j into this formula and reducing, we get

$$M_j + M_{j+1} = \frac{2(y_{j+1} - y_j)}{x_{j+1} - x_j} .$$

There are $n-1$ such equations since j runs from 1 to $n-1$. Note that the right hand side is completely known. There are $n-1$ equations to solve for n unknowns $M_1, M_2, \ldots, M_n$. One has the freedom to choose M_1. Then $M_2, M_3, \ldots, M_n$ are determined by

$$M_2 = \frac{2(y_2 - y_1)}{x_2 - x_1} - M_1 ,$$

$$M_3 = \frac{2(y_3 - y_2)}{x_3 - x_2} - M_2 ,$$

.....

How do we plot the spline which we have found?

In other words, how do we plot the parabolae $s_1, s_2, \ldots, s_{n-1}$ on the same graph?

The idea is to run a 'for' loop to plot each curve s_j individually while using the command 'hold on' to make sure that all plots are drawn on the same graph. For example,

```
for j = 1 : n-1
    u = x_j : 0.01 : x_{j+1} ;
    s = a_j * u.^2 + b_j * u + c_j ;
    plot (u,s);
    hold on
end
```