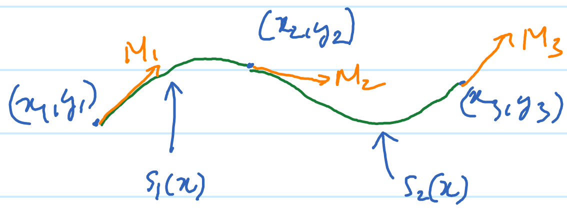
Lecture 23

Monday, March 2, 2020

How to find a quadratic spline curve that pass through n given points? Let us consider the simple case $n=3$.



Between (x_1,y_1) and (x_2,y_2) is the parabola $s_1(x)=a_1x^2+b_1x+c_1$.
Between (x_2,y_2) and (x_3,y_3) is the parabola $s_2(x)=a_2x^2+b_2x+c_2$.
We are to find these 6 coefficients: $a_1, b_1, c_1, a_2, b_2, c_2$.
The constraints are

$$\begin{cases} s_1(x_1)=y_1, \quad s_1(x_2)=y_2, \\ s_2(x_2)=y_2, \quad s_2(x_3)=y_3, \\ s_1'(x_2)=s_2'(x_2). \end{cases}$$

These 6 coefficients are "local" parameters because they are parameters of each parabola segment. We will express them in terms of "global" parameters M_1, M_2, M_3. Here M_1 is the slope of the spline curve (the concatenation of two parabolae) at the first point. M_2 is the slope at the second point, M_3 the third point.

First, we will express a_1, b_1, c_1 in terms of M_1 and M_2. This is done thanks to 3 equations:

$$\left.\begin{aligned} s_1'(x_1) &= M_1, \\ s_1'(x_2) &= M_2, \end{aligned}\right\} \text{solve for } a_1, b_1$$

$$s(x_1) = y_1. \quad \} \text{ solve for } c_1$$

The first two equations can be written as

$$\begin{cases} 2a_1x_1+b_1=M_1, \\ 2a_1x_2+b_1=M_2. \end{cases}$$

Viewing a_1 and b_1 as unknowns, one can solve for them using elimination / substitution / Cramer's rule / ...

$$a_1 = \frac{M_1 - M_2}{2(x_1 - x_2)}$$

$$b_1 = \frac{x_1 M_2 - x_2 M_1}{x_1 - x_2}$$

Then the third equation helps us solve for c_1:

$$y_1 = a_1 x_1^2 + b_1 x_1 + c_1$$

Thus, $c_1 = y_1 - a_1 x_1^2 - b_1 x_1$.

Then the parabola $s_1(x)$ can be written in terms of M_1 and M_2:

$$\begin{aligned} s_1(x) &= a_1 x^2 + b_1 x + c_1 \\ &= a_1 x^2 + b_1 x + y_1 - a_1 x_1^2 - b_1 x_1 \\ &= a_1 (x^2 - x_1^2) + b_1 (x - x_1) + y_1 . \end{aligned}$$

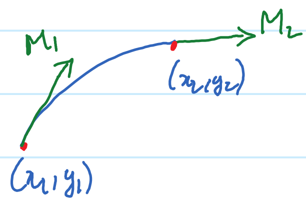
$$s_1(x) = \frac{M_1 - M_2}{2(x_1 - x_2)} (x^2 - x_1^2) + \frac{x_1 M_2 - x_2 M_1}{x_1 - x_2} (x - x_1) + y_1 . \qquad (*)$$

This formula says that whenever we know M_1 and M_2, we have an explicit formula of s_1 as a function of x. Similarly, whenever we know M_2 and M_3, we have an explicit formula of s_2 as a function of x.

$$s_2(x) = \frac{M_2 - M_3}{2(x_2 - x_3)} (x^2 - x_2^2) + \frac{x_2 M_3 - x_3 M_2}{x_2 - x_3} (x - x_2) + y_2 .$$

How can we determine M_1, M_2, M_3 ?

Recall that to derive the above formula for $s_1(x)$, we use the facts



$$\begin{cases} s_1 \text{ has slope } M_1 \text{ and } M_2 \text{ at the endpoints,} \\ s_1 \text{ passes through the left point, i.e. } (x_1, y_1). \end{cases}$$

We haven't imposed the condition that s_1 must pass through the right point yet. It is the condition $s_1(x_2) = y_2$. Substituting x by x_2 in equation $(*)$ we get

$$\underbrace{S_1(x_2)}_{y_2} = \frac{M_1 - M_2}{2(x_1-x_2)}(x_2^2 - x_1^2) + \frac{x_1 M_2 - x_2 M_1}{x_1 - x_2}(x_2 - x_1) + y_1.$$

$$= -\frac{M_1 - M_2}{2}(x_1 + x_2) - (x_1 M_2 - x_2 M_1) + y_1.$$

$$= M_1\left(\frac{x_2 - x_1}{2}\right) + M_2\left(\frac{x_2 - x_1}{2}\right) + y_1.$$

This gives us an equation that M_1 and M_2 have to satisfy:

$$(M_1 + M_2)\,\frac{x_2 - x_1}{2} = y_2 - y_1 \qquad (**)$$

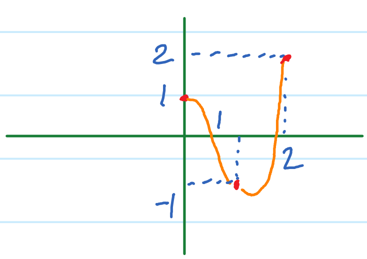
M_2 and M_3 also satisfy a similar equation:

$$(M_2 + M_3)\,\frac{x_3 - x_2}{2} = y_3 - y_2 \qquad (***)$$

Note that there are 3 unknowns: M_1, M_2, M_3. But we now have only 2 constraints. That means the system may have infinitely many solutions. We have the freedom to choose the value of one of M_1, M_2, M_3. Say $M_1 = 0$ (or some other number). Then one can solve for M_2 from (**) and then M_3 from (***).

The fact that M_1 can be chosen arbitrarily suggests that there are infinitely many ways to draw a quadratic spline that fits 3 given points.

Ex: Find a quadratic spline that passes through three points $(0,1), (1,-1), (2,2)$.



We are given $x_1, y_1, x_2, y_2, x_3, y_3$. Let M_1, M_2, M_3 be the slope of the curve at the first, second, third point.

Then the equations that M_1, M_2, M_3 have to satisfy are

$$\begin{cases} M_1 + M_2 = -4 \\ M_2 + M_3 = 6 \end{cases}$$

With $M_1 = 0$, we get $M_2 = -4$ and $M_3 = 10$. Plugging these numbers into (*), we get an explicit formula for S_1 as a function of x. Similarly, we also get an explicit formula for S_2 as a function of x.