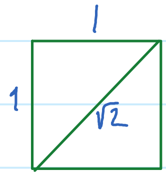
Lecture 2

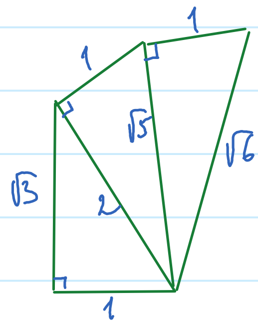
Wednesday, January 8, 2020

Some irrational numbers, such as $\sqrt{2}$ and π, arise naturally from real life. A unit square (square with side length equal to 1) has diagonal $\sqrt{2}$ by Pythagorean theorem.



One can find an approximate value of $\sqrt{2}$ by measuring the length of the diagonal of this square.

One can manage to evaluate $\sqrt{3}, \sqrt{5}, \sqrt{6}$ geometrically as follows. Credits perhaps go to the ancient Greek.



This approach is usually ad-hoc and relies heavily on the measuring instrument. For example, a tape with equally spaced marks by 1 mm can't be used to measure up to $1/100$ mm precision. Modern mathematics helps us compute, say $\sqrt{3}$, which any desired precision.

How do we compute $\sqrt{3}$?

It becomes necessary to clarify the term compute. In most cases, we are interested in the numerical value in decimal system. We know how to add two (or finitely many) numbers, subtract, multiply, divide. There are known algorithms to perform these tasks. Such algorithms can be programmed on a computer.

These four operations are also the only operations we know how to do by hand. So the problem becomes how to compute the numerical value of $\sqrt{3}$ by using only addition, subtraction, multiplication and division. On the other hand, $\sqrt{3}$ is an irrational number. Its representation in decimal system requires infinitely many digits with no periodic patterns. We, even computers, cannot find all of these digit in a finite amount of time. There is a natural need for approximation

One can phrase the problem this way: given a small number ε, called error tolerance, find an approximate value b of $\sqrt{3}$ (using addition, subtraction, multiplication, division only) such that $|b-\sqrt{3}|<\varepsilon$.

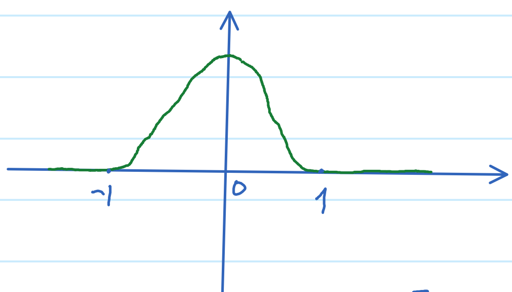
This can be done by Taylor expansion.

* Taylor's theorem (~1715)

For a large class of smooth functions $f:(a,b)\to\mathbb{R}$, we have

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots$$

Note that not all smooth functions have this property Consider the following example:



$f(x)$ is a bump function (smooth, equal to zero outside of the interval $[-1,1]$, nonzero inside).

For any $x_0<-1$ or $x_0>1$, we have

$$f(x_0) = f'(x_0) = f''(x_0) = \ldots = f^{(n)}(x_0) = \ldots = 0.$$

Thus, the power series is equal to zero for all x. We see that it cannot be equal to f for all x.

The class of functions f so that Taylor's theorem is true is called analytic functions.

What can we say about a general smooth function (not necessarily analytic)? One can always write

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{p_n(x)} + R_n(x)$$

where p_n is called the n'th Taylor polynomial and R_n the remainder.

Usually, p_n is of degree n, but that is not always the case. This is because the coefficient $f^{(n)}(x_0)$ can be equal to zero. For example,

$$f(x) = \sin x, \quad x_0 = 0,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

We have

$$p_0(x) = 0 \qquad (p_0 \text{ is always a constant polynomial})$$

$$p_1(x) = 0 + x = x$$

$$p_2(x) = 0 + x + 0 = x \qquad (\text{note that } p_2 \text{ is of degree } 1)$$

$$p_3(x) = 0 + x + 0 - \frac{x^3}{3!} = x - \frac{x^3}{6}$$

$$\dots$$

Taylor's approximation is that $f(x) \approx p_n(x)$ How good this appr. is depends on how small the error term $R_n(x)$ is. Taylor approximation is a method to approximate an arbitrary function by a polynomial A polynomial only involves addition and multiplication And one can evaluate the value of a polynomial by hand!

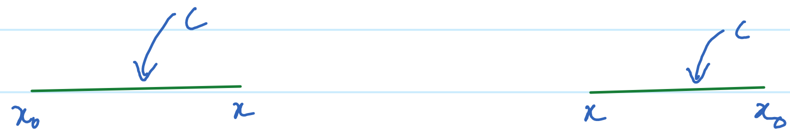
* Lagrange's theorem (~ 1772)

The error term has the following form:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

where c is a number between x_0 and x.

Note that c depends on x. Also, it can be bigger or smaller than x, depending on whether $x < x_0$ or $x_0 < x$.



Let's go back to compute $\sqrt{3}$. In other words, we want to evaluate $\sqrt{3}$ within some error tolerance, say $\varepsilon = 10^{-6}$ We know that

$$\sqrt{1} = 1, \quad \sqrt{4} = 2, \quad \sqrt{9} = 3$$

Let's consider the function $f(x) = \sqrt{x}$ We will try to approximate $f(3)$ based on what we know about f at $x_0 = 4$. Taylor expansion at $x_0 = 4$ gives

$$f(x) = p_n(x) + R_n(x)$$

where

$$p_n(x) = f(4) + \frac{f'(4)}{1!}(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \cdots + \frac{f^{(n)}(4)}{n!}(x-4)^n$$

Now substitute $x = 3$:

$$p_n(3) = f(4) + \frac{f'(4)}{1!}(-1) + \frac{f''(4)}{2!}1^2 + \cdots + \frac{f^{(n)}(4)}{n!}(-1)^n.$$

We have an approximation $\sqrt{3} = f(3) \approx p_n(3)$ with error $R_n(3)$.

There remain two tasks to do:

1. Compute $p_n(3)$ (make sure that it is possible to do by hand).
2. Make sure that $|R_n(3)| < \varepsilon = 10^{-6}$.

To compute $p_n(3)$, one need to compute $f(4), f'(4), f''(4), \ldots$

$$f(x) = x^{1/2}$$

$$f'(x) = \frac{1}{2} x^{-1/2}$$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right) x^{-3/2}$$

$$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) x^{-5/2}$$

$$\cdots$$

In general,

$$f^{(n)}(x) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(n-1)\right)x^{\frac{1}{2}-n}$$

Thus,

$$f^{(n)}(4) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(n-1)\right)\underbrace{4^{\frac{1}{2}-n}}_{=2^{1-2n}}$$

For each n, this number can be computed by hand. The remaining task is to make sure that the error term is under 10^{-6}.

$$|R_n(3)| < 10^{-6}$$

This is done by choosing sufficiently large n. We'll continue next time.