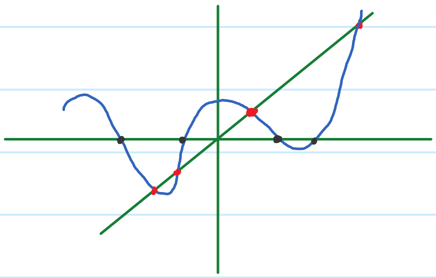
Lecture 16

Friday, February 14, 2020

With the bisection method and the Newton's method, we are able to find an approximate root of the equation $f(x)=0$. This is a rootfinding problem. There is "sister" problem of rootfinding problem, know as fixed point problem.

- x is a root of f if $f(x)=0$.
- x is a fixed point of f if $f(x)=x$.



The intersection of the graph of f and the x-axis are the roots of f (the black dots).
The intersection of the graph of f and the line $y=x$ are the fixed points of f (the red dots).

A fixed point problem can be converted into a rootfinding problem, for example,

$$f(x)-x=0,$$

$$\frac{f(x)}{x}-1=0,$$

$$f(x)^2-x^2=0,$$

$$\cdots$$

Likewise, a root finding problem can be converted into a fixed point problem, for example

$$f(x)+x=x,$$

$$x(f(x)+1)=x,$$

$$x-\frac{f(x)}{f'(x)}=x, \qquad \text{(Newton's method)}$$

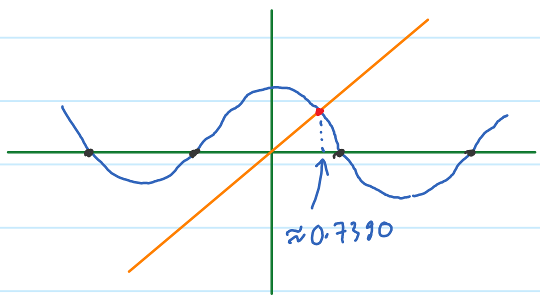
$$\cdots$$

Ex: Let us do the following experiment on the calculator:
Press $\boxed{1}$, then press $\boxed{=}$. Then press $\boxed{\cos}$ $\boxed{\text{ANS}}$ Then press $\boxed{=}$ many times. We get a new number each time we press the equal sign. After a number of times, we get a steady result, which is about 0.7390.

Let us call this number $a = 0.7390\ldots$ After pressing the equal sign, we should get $\cos a$. What is shown on the screen is a. Thus,

$$\cos a = a.$$

In other words, we have found a fixed point of the function cosine.



Let us take a closer look: we start with the initial value $x_0 = 1$. Then we get $x_1 = \cos x_0$ after pressing $\boxed{=}$. Then we get $x_2 = \cos x_1$, then $x_3 = \cos x_2, \ldots$. In general, we get a sequence (x_n) which is defined recursively as follows:

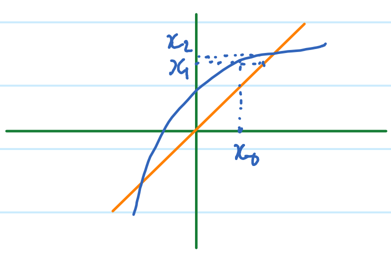
$$\begin{cases} x_0 = 1, \\ x_{n+1} = \cos x_n. \end{cases}$$

If this sequence has a limit, say $x = \lim x_n$. Then by taking the limit of both sides of the equation $x_{n+1} = \cos x_n$, we get

$$x = \cos x.$$

This confirms the experimental observation that x_n converges to a fixed point of $f(x) = \cos x$

The fixed point method has a very elegant illustration called cobweb diagram.



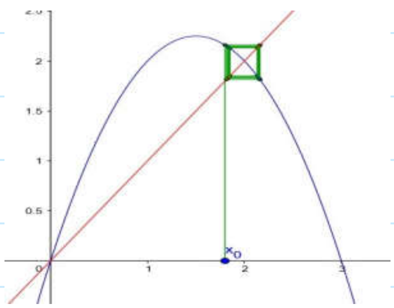
Starting from x_0, we draw a vertical line. It intersects the graph of f at exactly one point. The y-coordinate of this point is x_1. Then from this intersection point, draw a horizontal line. This line intersects the line $y=x$ at exactly one point. The x-coordinate of this point is x_1. Then from this intersection point, draw a vertical line. The intersection of this line and the graph of f has y-coordinate equal to x_2. Then from the intersection point, draw a horizontal line. The intersection of this line and the line $y=x$ has x-coordinate equal to x_2. We repeat this process.

One can experiment the fixed point method using cobweb diagram. A helpful applet can be found at the website:

https://www.geogebra.org/m/QJ79IWCL

Ex:

The function $f(x) = 3x - x^2$ has two fixed points $x=0$ and $x=2$.



The recursive sequence is

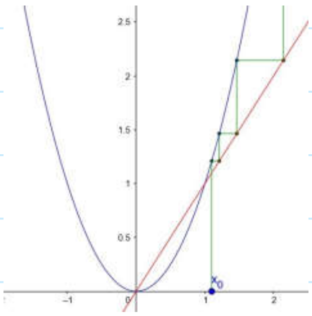
$$x_{n+1} = f(x_n) = 3x_n - x_n^2.$$

With $x_0 = 1.8$, we see that the sequence x_n converges (very slowly) to 2

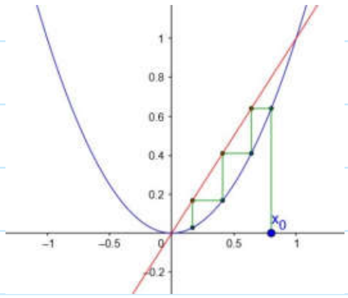
Ex. $f(x) = x^2$

This function has two fixed points: $x=0$ and $x=1$

If we choose $x_0 > 1$, we will get a divergence sequence. To be more precise, the sequence goes to infinity as $n \to \infty$. In a loose sense, one can say that this "limit" (the infinity) is a fixed point of f because $f(\infty) = \infty$



If we choose $0 < x_0 < 1$ then the sequence x_n will converge to 0.



It seems that the sequence will not converge to the fixed point 1 unless the initial point x_0 is chosen to be exactly 1.
In this case, 1 said to be an unstable fixed point, and 0 is said to be a stable fixed point.

A fixed point x^* is called stable if any choice of x_0 sufficiently near x^* will yield a sequence that converges to x^*.

Ex:

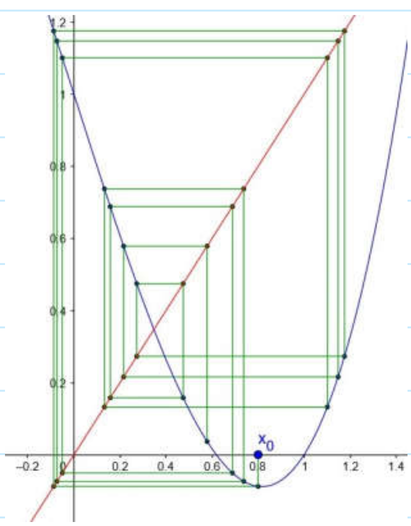
Solve for a root of $f(x) = x^3 - 3x + 1$ using fixed point method.
There are many ways to convert the rootfinding problem

$$x^3 - 3x + 1 = 0 \qquad (*)$$

into a fixed point problem. One way is to rewrite the equation as

$$\underbrace{x^3 - 2x + 1}_{g(x)} = x$$

The roots of f are the fixed points of g. By experimenting on the cobweb plotter, one can observe chaotic behavior of the sequence (x_n).

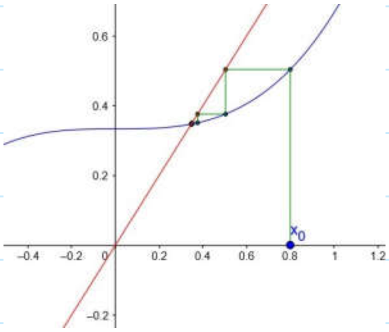


In other words, the sequence doesn't seem to converge. The choice of g is not good.

[Cobweb diagram of g with $x_0 = 0.8$]

Equation (*) can be written as

$$x^3+1 = 3x \quad \text{or} \quad \underbrace{\frac{1}{3}(x^3+1)}_{h(x)} = x.$$



We see that the sequence x_n converges quickly to a fixed point. Thus, the choice of h is better than g