Lecture 14

Friday, February 7, 2020

We have learn how to use bisection method and Newton's method to solve approximately the equation $f(x)=0$. Here x is a real number. In reality, we often solve for many unknowns at once. To solve for unknowns x and y, we typically need a system of two equations (since a single equation $f(x,y)=0$ usually gives infinitely many solutions, making the problem not meaningful). One can always bring a system of two equations and two unknowns into the form

$$\begin{cases} f(x,y)=0, \\ g(x,y)=0 \end{cases} \qquad (I)$$

How to solve numerically this system?

- Bisection method:

a ——— b

$f(a)f(b)<0$

The idea of bisection method is to find a region that contains a root. Then keep narrowing this region, thereby cornering the root. Two important ingredients in this method are:

1) An indicator that tells us that there is a root in a region. In this case, the indicator is the product $f(a)f(b)$. If this product is negative then we know for sure that there is a root on the interval $[a,b]$. An indicator is a compact, helpful piece of information that is easy to compute.

2) A method to narrow down the region. In this case, we narrow the interval $[a,b]$ by picking one half of this interval.

Let us discuss only the first issue. If the system (I) is a linear system, say

$$\begin{cases} ax+by+C_1=0 \\ cx+dy+C_2=0 \end{cases}$$

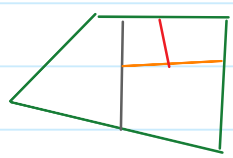
then it can be written in form of matrix

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{O}$$

If $\det A \neq 0$ then we know that $X = -A^{-1}C$. The system has a unique solution. In this situation, $\det A$ plays the role as an indicator to tell us whether there is a solution It is quite simple to compute.

How about the general case when f and g are not linear? There is also an indicator for a general function $F: \mathbb{R}^n \to \mathbb{R}^n$ that tells us whether F has a root in $\mathbb{R}^n$. In this case, $F = (f, g)$ and $n = 2$. This indicator is known as topological degree (introduced by Brouwer in 1911).

In 1976, Harvey and Stenger found a version of bisection method for two variables. In short, bisection method in 2D can be done, but is much more complicated than in 1D



- Newton's method.

Unlike the bisection method which relies heavily on geometry and is dimension sensitive, Newton's method is more robust in term of dimension change Remember that Newton's method is based on linearization of a function. In particular, x_{n+1} is constructed by linearizing function f at x_n.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

For 2D (or higher dimension), this idea still works:

f is to be replace by a vector function $F = (f, g)$,

f' is to be replace by DF (the derivative matrix of F).

To be more specific, $X_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$

$$DF(X_n) = \begin{bmatrix} \frac{\partial f}{\partial x}(X_n) & \frac{\partial f}{\partial y}(X_n) \\ \frac{\partial g}{\partial x}(X_n) & \frac{\partial g}{\partial y}(X_n) \end{bmatrix}$$ called derivative matrix, or Jacobian matrix.

$$F(X_n) = \begin{bmatrix} f(X_n) \\ g(X_n) \end{bmatrix} = \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}$$

The iteration formula of the Newton's method is

$$X_{n+1} = X_n - \underbrace{[DF(X_n)]^{-1}}_{\substack{\text{matrix} \\ (2\times 2)}} \underbrace{F(X_n)}_{\substack{\text{vector} \\ (2\times 1)}}$$

We will consider some example after the midterm.