Lecture 12

Monday, February 3, 2020

We know that Newton's method doesn't always guarantee success (i.e. the sequence x_n doesn't always converge.) However, it converges very quickly once it does converge. Under the assumption that the sequence x_n defined by

$$x_{n+1} = x_n - \frac{f'(x_n)}{f(x_n)}$$

converges to some α, we showed that $e_{n+1} \lesssim M e_n^2$ where

$$e_{n+1} = |x_{n+1} - \alpha| \quad \text{(error at the } n+1\text{'st step)},$$

$$e_n = |x_n - \alpha| \quad \text{(error at the } n\text{'th step)},$$

$$M = \frac{|f''(\alpha)|}{2|f'(\alpha)|}$$

The assumption that $x_n \to \alpha$ as $n \to \infty$ amounts to the assumption that $e_n \to 0$ as $n \to \infty$ It is fair to ask how fast e_n approaches 0. In other words, how do we quantify the speed of convergence of a sequence to its limit?

There are more than one way to quantify the speed of convergence. One of them is called the order of convergence

* Definition:

Suppose $e_{n+1} \lesssim C e_n^p$ for some $C > 0$, $p \geq 1$ (for all large n) then the sequence (x_n) is said to converge to α with the order of convergence p. If $p=1$ then C is said to be the linear rate of convergence.

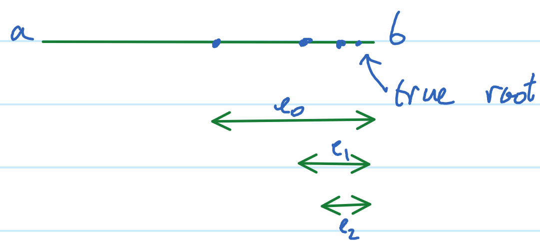
Here $e_n = |x_n - \alpha|$ and $e_{n+1} = |x_{n+1} - \alpha|$ The larger p is, the faster the convergence. We have seen that Newton's method has order of convergence $p=2$. When $p>1$, the constant C is usually not interesting. But if $p=1$, C becomes relevant. Let us take a look at the estimate

$$e_{n+1} \lesssim C e_n.$$

If $C=1$ then the estimate becomes $e_{n+1} \lesssim e_n$, which says that e_{n+1} is at most e_n. This doesn't provide any information on how fast e_n goes to zero. In general, the estimate $e_{n+1} \lesssim Ce_n$ where $C \gtrsim 1$ is not useful. However, when $C < 1$, it becomes useful.

For example, consider the estimate $e_{n+1} \lesssim \frac{1}{2} e_n$ says that after each step the error is reduced almost by one half time. Thus, doing 100 iterations will reduce the initial error (i.e e_0) by $\frac{1}{2^{100}}$ times.

Consider the bisection method. It is conceivable that the error is worst when to true root is close to one end of the interval $[a, b]$.



In this scenario, the error is almost reduce by one half time after each iteration: $e_{n+1} \lesssim \frac{1}{2} e_n$.
Thus, the bisection method has order of convergence 2, and linear rate of convergence equal to 1/2.

Ex: $$x_{n+1} = \frac{x_n}{2} + \frac{1}{2x_n}, \qquad x_0 = 2.$$

By calculator, we have $x_1 = 1.25$, $x_2 = 1.025$, $x_3 = 1.000304\ldots$, $x_4 = 1.000000046\ldots$

We guest the limit of x_n is 1 How can we be sure that it is 1, not $1.0000\ldots01$? Let us put $a = \lim x_n$. Take the limit of both sides of the iteration formula $$x_{n+1} = \frac{x_n}{2} + \frac{1}{2x_n}$$

we get $$a = \frac{a}{2} + \frac{1}{2a}$$

This equation is equivalent to $a^2 = 1$, which gives $a = 1$ or -1. Because $x_n > 0$ for all n, the limit cannot be negative. Thus, $\lim x_n = 1$

How fast does x_n approach 1? We see that x_n approaches 1 very fast.

$$e_0 = |x_0 - 1| = |2-1| = 1,$$
$$e_1 = |x_1 - 1| = 0.25$$
$$e_2 = |x_2 - 1| = 0.025$$
$$e_3 = |x_3 - 1| = 0.0003\ldots$$
$$e_4 = |x_4 - 1| = 0.00000004\ldots$$

The order of convergence should be greater than 1 because the quotient e_{n+1}/e_n decreases to 0 instead of being close to a constant $C>0$. Let us subtract 1 from the iteration formula:

$$x_{n+1} - 1 = \frac{x_n}{2} + \frac{1}{2x_n} - 1 = \frac{x_n^2 + 1 - 2x_n}{2x_n} = \frac{(x_n-1)^2}{2x_n}.$$

Take the absolute value of both sides:

$$|x_{n+1} - 1| = \frac{|x_n - 1|^2}{2x_n}$$

or

$$e_{n+1} = \frac{e_n^2}{2x_n}.$$

When n is big, $x_n \approx 1$. Thus, $e_{n+1} \approx \dfrac{e_n^2}{2}$.

We conclude that the order of convergence is 2 It explains why x_n goes to 1 very fast