Lecture 11

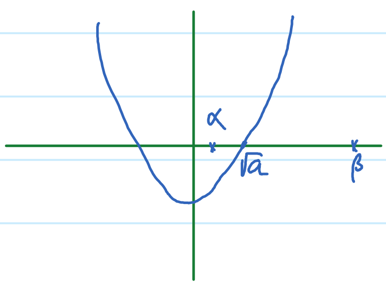
Friday, January 31, 2020

If we are to program the square root function for a calculator, which numerical method should we choose: bisection or Newton's method?

Observation: we don't know what number the user will enter, except that it is a positive number within the range the calculator permits. Let $a > 0$ be the input. We want to give a numerical value of $\sqrt{a}$. This is to find the positive root of the polynomial

$$f(x) = x^2 - a.$$

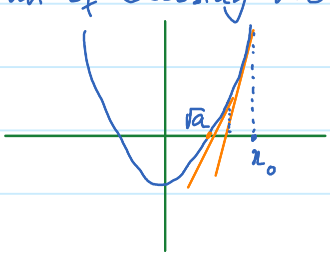
Suppose we choose the bisection method. What we need to do is to find α, β such that $f(\alpha)$ and $f(\beta)$ have different size. Because we

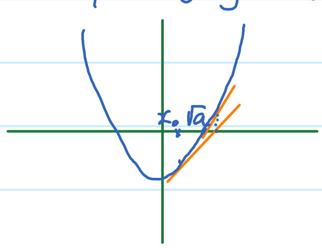


don't know what a is, we should choose α very small, and β very large. To make sure that our choice of α, β should work for any input a, we can choose $\alpha = 0$ (so that $f(\alpha) = -a < 0$) and β = the largest number that can be represented by the calculator.

Then after each step of the bisection method, the interval of inspection (starting with $[a_0, b_0] = [\alpha, \beta]$) keeps reducing by half. It will take quite many steps to get a good approximation of the root. Also, at each step, one needs to find the signs of $f(a_n)$ and $f(b_n)$. This will take costly calculations. Hence, the bisection method, although guarantees success, will take a while to compute. This is not preferred, especially on a pocket calculator.

Let's consider Newton's method (attribute to Newton and Raphson, 1680s): instead of choosing two points to start with, we only need to choose one.





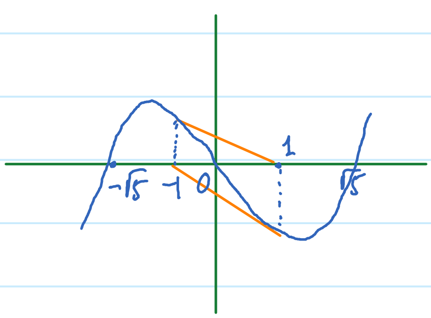
The iteration formula is
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^2 - a}{2x_n}$$
$$= \frac{x_n}{2} + \frac{a}{2x_n}$$

When a is given, the iteration formula is completely well-defined. It only involves a multiplication, two divisions, and one addition. These operations can be done by the calculator. We also see on the picture that no matter where x_0 (as long as $x_0 > 0$), the sequence $x_0, x_1, x_2, \ldots$ always converges to $\sqrt{a}$. The convergence is rapid. Recall the example last time: it took only 3 iterations to get a good approximation of $\sqrt{3}$.

In general, Newton's method doesn't guarantee success for any choice of initial point x_0. In this problem, fortunately the sequence indeed converges to $\sqrt{a}$ regardless of the choice of x_0. One can simply choose $x_0 = 1$ for any input a from the user.

In conclusion, Newton's method is more efficient in this problem.

Ex: Estimate roots of polynomial $f(x) = x^3 - 5x$ by Newton's method using $x_0 = 1$



We have $f'(x) = 3x^2 - 5$

The iteration formula of Newton's method is
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^3 - 5x_n}{3x_n^2 - 5}$$
$$= \frac{2x_n^3}{3x_n^2 - 5}$$

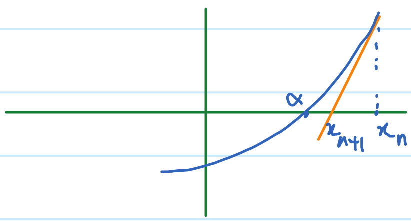
Since $x_0 = 1$, we get $x_1 = -1$, $x_2 = 1$, $x_3 = -1$, $x_4 = 1$, ...

This is an example showing that Newton's method doesn't always guarantee success.

One can experiment Newton's method visually by using an applet on this website : https://www.geogebra.org/m/DGFGBJyU

We have seen from the last lecture that the Newton's method converges very quickly

It is fair to ask : how fast does the Newton method converge, assuming that it does converge?



Let α be a true root of $f(x)$. We want to see how fast x_n (obtained from Newton's method) approaches α.

We have $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

We see that x_{n+1} is the intersection of the x-axis and the tangent line of the graph of f, while α is the intersection of the x-axis and the graph of f. By saying $x_{n+1} \approx \alpha$, we are approximating the graph of f by its tangent line. We are virtually using linear approximation of f.

The first Taylor approximation of f about α is

$$\begin{aligned} f(x) &= p_1(x) + R_1(x) \\ &= \underbrace{f(\alpha)}_{=0} + f'(\alpha)(x-\alpha) + R_1(x) \\ &= f'(\alpha)(x-\alpha) + R_1(x) \end{aligned}$$

Lagrange's theorem says that $$R_1(x) = \frac{f''(c)}{2!}(x-\alpha)^2$$

for some c in between α and x. Thus,

$$f(x) = f'(\alpha)(x-\alpha) + \frac{f''(c)}{2!}(x-\alpha)^2$$

Now substitute x by x_n. Since c will now be in between x_n and α, we should write it as c_n to indicate the dependence of c on n.

We get $f(x_n) = f'(\alpha)(x_n - \alpha) + \frac{f''(c_n)}{2}(x_n-\alpha)^2$.

Divide both sides by $f'(x_n)$:

$$\frac{f(x_n)}{f'(x_n)} = \frac{f'(\alpha)(x_n-\alpha)}{f'(x_n)} + \frac{f''(c_n)}{2f'(x_n)}(x_n-\alpha)^2$$

Using this equality, we can rewrite x_{n+1} as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{f'(\alpha)}{f'(x_n)}(x_n-\alpha) - \frac{f''(c_n)}{2f'(x_n)}(x_n-\alpha)^2.$$

Subtract α from both sides:

$$x_{n+1} - \alpha = (x_n - \alpha) - \frac{f'(\alpha)}{f'(x_n)}(x_n-\alpha) - \frac{f''(c_n)}{2f'(x_n)}(x_n-\alpha)^2$$

$$= (x_n - \alpha)\left(1 - \frac{f'(\alpha)}{f'(x_n)}\right) - \frac{f''(c_n)}{2f'(x_n)}(x_n-\alpha)^2. \qquad (*)$$

we then use 0'th order Taylor approximation for $f'(x)$ about α.

$$f'(x) = f'(\alpha) + f''(d)(x-\alpha)$$

where d is in between x and α. With $x = x_n$, we have

$$f'(x_n) = f'(\alpha) + f''(d_n)(x_n - \alpha)$$

for some d_n in between x_n and α. Divide both sides by $f'(x_n)$:

$$1 = \frac{f'(\alpha)}{f'(x_n)} + \frac{f''(d_n)}{f'(x_n)}(x_n-\alpha).$$

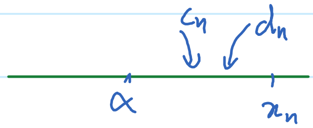
Hence, $$1 - \frac{f'(\alpha)}{f'(x_n)} = \frac{f''(d_n)}{f'(x_n)}(x_n-\alpha).$$

Now substitute this equality into $(*)$:

$$x_{n+1} - \alpha = \frac{f''(d_n)}{f'(x_n)}(x_n-\alpha)^2 - \frac{f''(c_n)}{2f'(x_n)}(x_n-\alpha)^2$$

$$= \frac{2f''(d_n) - f''(c_n)}{2f'(x_n)}(x_n-\alpha)^2$$

Under the assumption that x_n converges to α, we see that c_n, d_n, x_n must be close to α when n is large.



Thus, it is reasonable to make the approximation

$$f''(c_n) \approx f''(\alpha),$$
$$f''(d_n) \approx f''(\alpha),$$
$$f'(x_n) \approx f'(\alpha).$$

Then

$$x_{n+1} - \alpha \approx \frac{f''(\alpha)}{2f'(\alpha)} (x_n - \alpha)^2$$

Now take the absolute value of both sides:

$$e_{n+1} \approx \frac{|f''(\alpha)|}{2|f'(\alpha)|} e_n^2 \qquad (**)$$

This relation suggests that e_n goes to 0 very quickly as $n \to \infty$.

Let's consider an example: $f(x) = x^2 - 3$.

This function has a root $\alpha = \sqrt{3}$. The iteration formula of Newton's method is

$$x_{n+1} = \frac{x_n}{2} + \frac{3}{2x_n}.$$

With the initial point $x_0 = 2$ for example, we can ask how fast x_n goes to $\sqrt{3}$ as $n \to \infty$. In this problem,

$$e_n = |x_n - \alpha| = |x_n - \sqrt{3}|,$$
$$f'(x) = 2x,$$
$$f''(x) = 2.$$

Then $(**)$ becomes $\quad e_{n+1} \approx \frac{\sqrt{3}}{2} e_n^2 < e_n^2.$

We have $e_0 = |x_0 - \sqrt{3}| = 2 - \sqrt{3} < \frac{1}{2}$. Then

$$e_1 \lesssim e_0^2 = \frac{1}{4},$$
$$e_2 \lesssim e_1^2 = \frac{1}{16},$$
$$e_3 \lesssim e_2^2 = \frac{1}{256}.$$
$$\cdots\cdots$$

One can perceive that e_n goes to 0 very rapidly.