Lecture 1

Monday, January 6, 2020

In this course, we will discuss the following topics:

1) Taylor approximation: one should have already been familiar with Taylor expansion / Taylor series. We will focus on how to apply Taylor approximation and estimate the error.

2) Error analysis: investigate the sources that lead to errors and how to control them

3) Root-finding problems: a lot of real life problem can be formulated as finding solution x from the equation $f(x) = 0$.

For example,

$$2x - 1 = 0, \qquad (1)$$

$$x^2 - 2x - 2 = 0, \qquad (2)$$

$$x^3 - x + 1 = 0. \qquad (3)$$

We know how to solve x from (1). $x = 1/2$

- Solving equation (1) leaded to the discovery of rational numbers ($\mathbb{Q}$) since the set of integer doesn't contain solutions to (1).
- Solving equation (2) leaded to the discovery of real numbers ($\mathbb{R}$).
- Solving equation (3) leaded to the discovery of complex numbers ($\mathbb{C}$) in 1545 by Italian mathematician G. Cardano (1545).

In this course, we will learn simple methods to find solutions approximately.

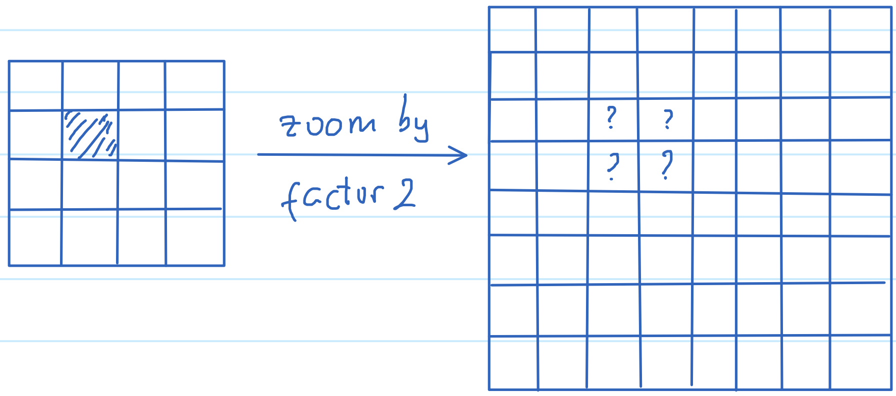
4) Interpolation problems.

Suppose we have a digital image of size 600×800 and that it is a gray-scale image. Each pixel of the image is assigned a number from 0 to 255 called gray scale.

0 .... black

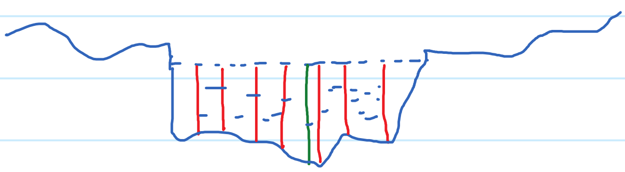
255 .... white

How do we magnify the image by a factor of two (i.e. to obtain an image of size 1200 x 1600) ?



Each pixel of the original image corresponds to 4 pixels of the new image. We need to assign a gray scale to each of these new pixels. Here we are facing an interpolation problem : the amount of data we desire is less than the amount of data available.

Another example of interpolation problem: suppose one wants to measure the depth of a lake. He can measure the depth of the lake at finitely many locations (the red lines). How about the other locations which are not sampled?



We will investigate the interpolation problem in detail later.

5) Numerical integration :

For some functions, one can find an explicit form of antiderivative, making it easy to evaluate definite integral. For example, the function $f(x) = x + 1/x$ has antiderivative $F(x) = \frac{x^2}{2} + \ln|x|$. Thus,

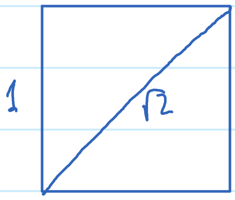
$$\int_1^2 f(x)\,dx = F(2) - F(1) = \cdots$$

There are other functions which do not have an elementary anti-derivative (i.e. antiderivative not written in term of the elementary functions such as polynomials, exponential, logarithm, trigonometric.)

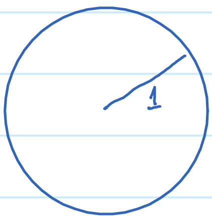
Numerical integration is a family of numerical method that helps us find an approximate value of an integral without knowing the antiderivative. A method we already knew is Riemann integration

Historically, numerical integration methods appear very early, for example from the need of measuring area, length, volume, ...

For example, what is the length of the diagonal of a square of side length 1 ?

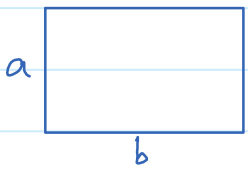


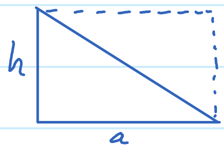
What is the area of a circle of radius 1 ?



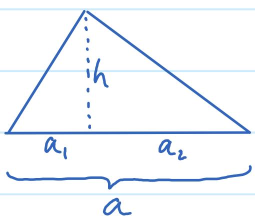
Integral not only helps us compute the area of an object but also <u>define</u> the area.

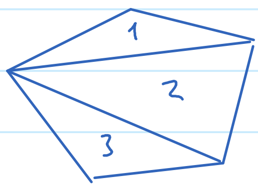
If a square of side length 1 is chosen as unit then the area of a rectangle of length $1/2$ and width $1/3$ will have area $1/6$ (since 6 of such rectangles fit into the square). The area of a general rectangle with length a and width b has area ab.

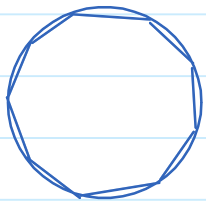




A right triangle has area $\frac{1}{2}ah$ because it is a half of a rectangle.

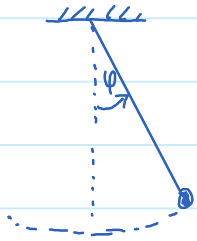






As early as 260s BC, Archemedes approximated the area of a circle by the area of regular polygons incribed in it.

If time allows, we will discuss how to use numerical methods to solve differential equations. D. E. are very common in applied sciences A classic example is the equation of a pendulum:



$$\varphi'' + a \sin \varphi = 0 .$$