Lecture 19 (2/27/2019)

Question: is it true that $\sum_{n=1}^{\infty} a_n b_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$?

No, take $a_n = 1$ (for all n) and $b_n = \frac{1}{2^n}$.

Even finite sum doesn't have this property, for example with $n=2$:

$$a_1 b_1 + a_2 b_2 \neq (a_1 + a_2)(b_1 + b_2)$$

Convergence tests: quick ways to check if a series converges or diverges.

"General term goes to zero" test:

- If $\sum a_k$ converges then $a_k \to 0$ as $n \to \infty$.

why? $S_n = a_1 + a_2 + \cdots + a_n$

the series converges $\Longleftrightarrow$ $\underbrace{S_n \text{ converges}}$

this implies that the increment of the sum goes to 0.

$S_{n+1} - S_n = a_{n+1}$: increment at n'th step

Therefore a_n must go to 0.

- If $a_n \not\to 0$ as $n \to \infty$ then $\sum a_k$ doesn't converge.

Ex:

$$\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$$

The general term of the series is $a_n = \frac{n^2}{n^2+1}$. This term is an increment of the sum at the $(n-1)$'st step.

$$a_n \longrightarrow 1 \neq 0 \quad \text{as } n \to \infty$$

Thus, the series diverges.

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+1}$ (alternating series)

General term $a_n = \frac{(-1)^n n^2}{n^2+1}$ doesn't converge to 0. Why?

$$|a_n| = \frac{n^2}{n^2+1} \to 1 \neq 0$$

The series diverges

* Warning: If $a_n \to 0$ as $n \to \infty$, it's not necessarily true that $\sum a_n$ converges.

For example, the series $\sum \frac{1}{\sqrt{k}}$ diverges because the partial sum

$$S_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1)$$

diverges (Homework 5).

<u>Comparison test</u>

- If $|a_n| \le b_n$ for all n sufficiently large $(n \geq N)$

and $\sum b_n$ converges then $\sum a_n$ also converges.

(A series dominated by a convergent series is convergent)

- If $0 \le a_n \le b_n$ for all n sufficiently large $(n \geq N)$

and $\sum a_n$ diverges then $\sum b_n$ also diverges.

(A series that dominates a divergent series with nonnegative terms is also divergent)

Ex:

$$\sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n}{n^2+1}}_{a_n}$$

$$|a_n| = \frac{1}{n^2+1} \leq \frac{1}{n^2} = b_n$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (showed in last lecture)

Therefore, the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ converges

Ex: $\sum_{n=3}^{\infty} \frac{1}{2^n - n^2}$

We notice that as n large, n^2 is dominated by 2^n, so the fraction $\frac{1}{2^n-n^2}$ is essentially $\frac{1}{2^n}$ as n large. Thus, we guess that the given series converges. How to prove?

Guess: $\frac{1}{2^n - n^2} < \frac{2}{2^n}$

This is equivalent to $2^n < 2(2^n - n^2)$, which is equiv. to $2n^2 < 2^n$, which is a true inequality for n sufficiently large.

Thus, $\sum_{n=3}^{\infty} \frac{1}{2^n-n^2}$ is dominated by $\sum_{n=3}^{\infty} \frac{2}{2^n}$, which is a convergent series.