Lecture 11 (2/1/2019)

* Linear dependence / independence

A set of vectors $v_1, v_2, \ldots, v_k$ is called linearly dependent if there exists a nontrivial linear combination that has sum 0.

In other words, $v_1, \ldots, v_k$ are linearly dependent if there are $c_1, c_2, \ldots, c_k$ not all zero such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0.$$

If $v_1, \ldots, v_k$ are not linearly dependent, they are said to be linearly independent.

Ex:

- $v_1 = (1,2)$, $v_2 = (2,4)$ are lin. dep. because $2v_1 - v_2 = 0$
- $v_1 = (1,2)$, $v_2 = (2,3)$ are lin. ind because the equation

$$c_1 v_1 + c_2 v_2 = 0$$

is equivalent to the system

$$\begin{cases} c_1 + 2c_2 = 0 \\ 2c_1 + 3c_2 = 0 \end{cases}$$

which only has solution $\underbrace{(c_1, c_2) = (0,0)}_{\text{called trivial solution}}$.

To check if the given vectors are lin. ind. is to check if a system of lin. eqs has only trivial solution.

The equation $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$ has matrix form

$$\underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{bmatrix}}_{\substack{A \\ \text{(coefficient matrix)}}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}}_{\substack{x \\ \text{(unknowns)}}} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}_{b}$$

$x=0$ is always a solution to $Ax=0$. In what situations is it the unique solution?

Augmented matrix of the system: $[A \mid 0]$

$$[A \mid 0] \xrightarrow{\text{row operations}} [B \mid 0] \quad \text{in REF}$$

Since the augmented column (which is 0) isn't changed by row operations, one can obmit this column before doing row operations.

$$A \xrightarrow{\text{row operations}} B \quad \text{in REF}$$

If B has a column that has no pivot entries then the system has infinitely many sol.

In this case $v_1, v_2, \dots, v_k$ are lin. dependent.

If every col. of B contains a pivot entry then the system has a unique sol.

In this case, $v_1, v_2, \dots, v_k$ are lin. independent.

Ex: $v_1 = (1,2,3)$
$v_2 = (2,1,0)$
$v_3 = (1,-1,-3)$

Are these vectors lin. dep. or ind.?

$$A = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 0 & -3 \end{bmatrix}$$

$$A \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix} \xrightarrow{R_3 = R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

↑ this col. has no pivot entries

Conclusion: v_1, v_2, v_3 are lin. dependent.

How to find a nontrivial linear combination that adds up to 0?

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

the third column has no pivot entries.

c_3 is a free variable

Take $c_3 = 1$ for example.

From the second row: $-3c_2 - 3c_3 = 0 \Rightarrow c_2 = -1$

From the first row: $c_1 + 2c_2 + c_3 = 0 \Rightarrow c_1 = -2c_2 - c_3 = 1$

Therefore,

$$\boxed{v_1 - v_2 + v_3 = 0}$$

Ex:

$v_1 = (1, 2, 3, 4)$

$v_2 = (2, 3, 1, 2)$ Check if these vectors are lin. dep. or ind.

$v_3 = (2, 0, -1, 1)$

$$A = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 3 & 1 & -1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$A \xrightarrow{\text{row operations}} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -4 \\ 0 & 0 & 13 \\ 0 & 0 & 0 \end{bmatrix}$$

Every col. has a pivot entry.

v_1, v_2, v_3 are linearly independent.

* If the number of vectors is equal to the length of each vectors: ($v_1, v_2, \dots, v_n \in \mathbb{R}^n$)

A is a square matrix.

The system $Ax = 0$ has a unique sol. if and only if A is invertible, which is equivalent to $\det A \neq 0$.

Ex:

$v_1 = (2, 0, -1)$

$v_2 = (3, 2, -2)$

$v_3 = (1, 0, 1)$

$$A = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 2 & 0 \\ -1 & -2 & 1 \end{bmatrix}$$

$\det A = 6 \neq 0$

v_1, v_2, v_3 are linearly independent.