Solution to some prob. in HW8

1. j) $\displaystyle \sum \frac{e^n+2^n}{3^n-2^n}$

analysis $\left\{ \begin{array}{l} \text{Since } 2<e<3, \quad 2^n \ll e^n \ll 3^n \text{ when } n \text{ large.} \\ \text{The term } \dfrac{e^n+2^n}{3^n-2^n} \text{ is essentially } \dfrac{e^n}{3^n} \text{ when } n \text{ large} \\ \displaystyle \sum \frac{e^n}{3^n} = \sum \left(\frac{e}{3}\right)^n \text{ converges because } 0<\frac{e}{3}<1 \end{array} \right.$

We make the above observation rigorous by trying to show that

$$\frac{e^n+2^n}{3^n-2^n} < 2\,\frac{e^n}{3^n} \quad \text{when } n \text{ large}$$

How?

Multiply both side by $\frac{3^n}{e^n}$:

$$\frac{e^n+2^n}{3^n-2^n}\,\frac{3^n}{e^n} \overset{?}{<} 2$$

$$LHS = \frac{1+\left(\frac{2}{e}\right)^n}{1-\left(\frac{2}{3}\right)^n} \longrightarrow 1 \quad \text{as } n\to\infty$$

Therefore $LHS < 2$ as n large.

3. b) $A = \begin{bmatrix} -2 & -2 & -9 \\ -1 & 1 & -3 \\ 1 & 1 & 4 \end{bmatrix}$

$$\det(A-\lambda I_3) = \begin{vmatrix} -2-\lambda & -2 & -9 \\ -1 & 1-\lambda & -3 \\ 1 & 1 & 4-\lambda \end{vmatrix} \begin{matrix} -2-\lambda & -2 \\ -1 & 1-\lambda \\ 1 & 1 \end{matrix}$$

$$= (-2-\lambda)(1-\lambda)(4-\lambda) + (-2)(-3)1 + (-9)(-1)1 - (-9)(1-\lambda)1 - 1(-3)(-2-\lambda) - (-1)(-2)(4-\lambda)$$

$$= -8 + 6\lambda + 3\lambda^2 - \lambda^3 + 6 + 9 + (9 - 9\lambda) - (6 + 3\lambda) - (8 - 2\lambda)$$

$$= -\lambda^3 + 3\lambda^2 - 4\lambda + 2$$

$$= (1-\lambda)(\lambda^2 - 2\lambda + 2)$$

This polynomial has three roots : $\lambda_1 = 1$, $\lambda_2 = 1+i$, $\lambda_3 = 1-i$

* Find eigenvectors corresponding to $\lambda_1 = 1$:

$$A - I_3 = \begin{bmatrix} -3 & -2 & -9 \\ -1 & 0 & -3 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow[\substack{R_2 = R_2 + R_3 \\ R_1 \Leftrightarrow R_1}]{R_1 = R_1 + 3R_3} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[R_3 = R_3 - R_2]{R_1 = R_1 - R_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

column without pivot entries

$x_3 = t$ (free variable)

$x_2 = 0$ (from 2^{nd} row)

$x_1 = -3t$ (from 1^{st} row)

$\Rightarrow$ $v_1 = (-3t, 0, t)$ for any t.

Pick $t = 1$, which gives $v_1 = (-3, 0, 1)$

* Find eigenvectors corresponding to $\lambda_2 = 1+i$:

$$A - (1+i)I_3 = \begin{bmatrix} -3-i & -2 & -9 \\ -1 & -i & -3 \\ 1 & 1 & 3-i \end{bmatrix} \xrightarrow[\substack{R_2 = R_2 + R_3 \\ R_1 \Leftrightarrow R_3}]{R_1 = R_1 + (3+i)R_3} \begin{bmatrix} 1 & 1 & 3-i \\ 0 & -i+1 & -i \\ 0 & 1+i & 1 \end{bmatrix}$$

$$\xrightarrow[\substack{R_1 = R_1 - R_2 \\ R_3 = R_3 - (1+i)R_2}]{R_2 = \frac{R_2}{-i+1}} \begin{bmatrix} 1 & 0 & \frac{5-i}{2} \\ 0 & 1 & \frac{1-i}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

col. without pivot entries

$$\begin{cases} x_3 = t & \text{(free variable)} \\ x_2 = -\frac{1-i}{2}t & \text{(by } 2^{nd} \text{ row)} \\ x_1 = -\frac{5-i}{2}t & \text{(by } 1^{st} \text{ row)} \end{cases}$$

Pick $t = -2$: $v_2 = (5-i, 1-i, -2)$

* Find eigenvectors corresponding to $\lambda_3 = 1-i$:

Because $\lambda_3 = \bar{\lambda}_2$ (complex conjugate), v_3 can be taken as the conjugate of v_2 :

$$v_3 = (5+i, 1+i, -2)$$