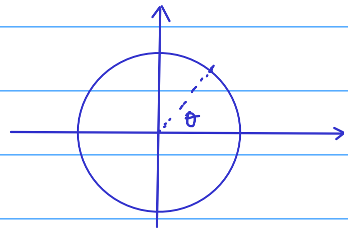
Lecture 9 (2/6/2019)

Inverse trigonometric functions:



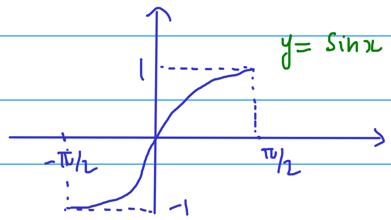
$\sin\theta = x \in [-1,1]$

there are infinitely many such θ's.

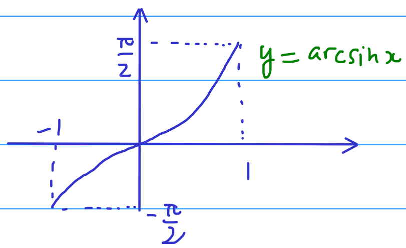
However, there is only one θ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$\arcsin : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the inverse of the sin function.

$\arcsin x = \theta \implies \sin\theta = x$.



take reflection about the line $y=x$

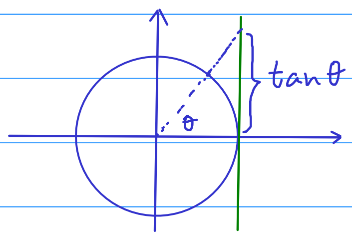


What is the derivative of arcsin?

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$ (a consequence of the chain rule)

One can define the inverse of the tangent function likewise:

$\tan\theta = x \in \mathbb{R}$

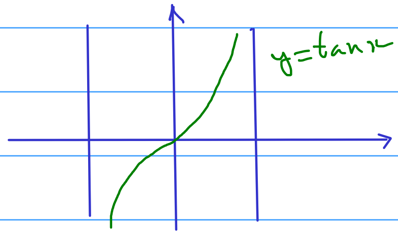


There are infinitely many such θ's.

There is only one θ in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

is the inverse function of $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$



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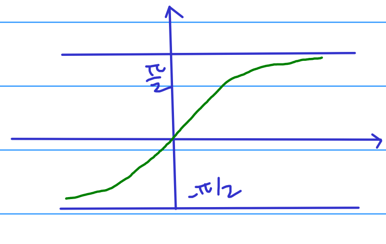


$$(\arctan x)' = \frac{1}{1+x^2}$$

Ex: Compute $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$

$$= \arctan x \Big|_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Ex:

Compute $\displaystyle\int_2^3 \frac{1}{4+9x^2}\,dx$

$$I = \frac{1}{4}\int_2^3 \frac{1}{1+\frac{9x^2}{4}}\,dx$$

Put $u = \frac{3x}{2}$.

$du = \frac{3}{2}\,dx$

x	2	3
u	3	9/2

$$I = \frac{1}{4}\int_3^{9/4} \frac{1}{1+u^2}\,\frac{2}{3}\,du$$

$$= \frac{1}{6}\arctan u \Big|_3^{9/4}$$

$$= \frac{1}{6}\left(\arctan\left(\frac{9}{4}\right) - \arctan 3\right)$$

Ex:

$$I = \int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$$

$u = \frac{x}{2}$

x	0	1
u	0	1/2

$du = \frac{1}{2}\,dx$

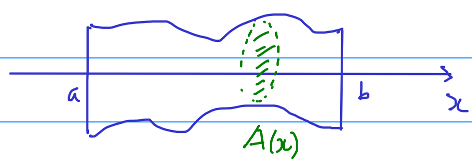
$$\frac{1}{\sqrt{4-x^2}}\,dx = \frac{1}{\sqrt{4-4u^2}}\,2du = \frac{1}{\sqrt{1-u^2}}\,du$$

$$I = \int_0^{1/2} \frac{1}{\sqrt{1-u^2}}\,du = \arcsin u \Big|_0^{1/2}$$

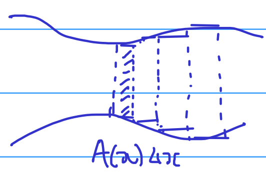
$$= \arcsin \tfrac{1}{2} - \arcsin 0$$

$$= \pi/6$$

*Compute volume by slicing:



Consider a solid that extends from $x = a$ and $x = b$. Suppose the cross section area at position x is $A(x)$.



Volume is approximated by

$$\sum A(x)\,\Delta x$$

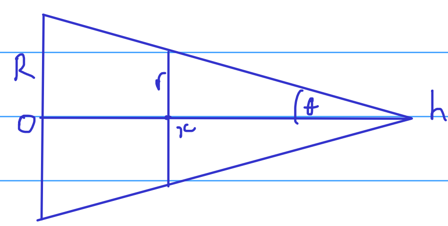
This is a Riemann sum of the function $A(x)$ on the interval $[a,b]$.

Thus, the exact volumn is

$$V = \int_a^b A(x)\,dx$$

Ex:

Find volume of the cone with height h and circular base of radius R.



Cross section at position x is a circle with radius r.

$$\tan\theta = \frac{R}{h} = \frac{r}{h-x}$$

Thus,
$$r = \frac{R}{h}(h-x)$$

$$A(x) = \pi r^2 = \frac{\pi R^2}{h^2}(h-x)^2$$

$$\text{Volume of the cone} = \int_0^h A(x)\,dx = \frac{\pi R^2}{h^2}\int_0^h (h-x)^2\,dx$$

$$= \frac{\pi R^2}{h^2}\left(-\frac{(h-x)^3}{3}\right)\Bigg|_0^h$$

$$= \frac{\pi R^2}{h^2}\;\frac{h^3}{3}$$

$$= \frac{\pi R^2 h}{3}$$