Lecture 8 (2/4/2019)

* Integral by substitution:
  - Indefinite integral:

$$\int g(u(x))\, u'(x)\, dx = \int g(u)\, du$$

  - Definite integral:

$$\int_a^b g(u(x))\, u'(x)\, dx = \int_{u(a)}^{u(b)} g(u)\, du$$

In practice, the substitution rule for indefinite integral and definite integral work the same way, except that one must update the limits of integral after changing dx to du.

Ex:

$$\int_{-1}^{2} \frac{x}{\sqrt{1+x^2}}\, dx$$

- Change of variable: $u = 1+x^2$

x	-1	2
u	2	5

- $du = u'dx = 2x\,dx$
- $f(x)dx = \frac{x}{\sqrt{u}}\, dx = \frac{1}{2}\frac{1}{\sqrt{u}}\, du$
- Integrate both sides:

$$\int_{-1}^{2} f(x)\, dx = \int_{2}^{5} \frac{1}{2}\frac{1}{\sqrt{u}}\, du = \frac{1}{2}\, 2\sqrt{u}\Big|_2^5 = \sqrt{5}-\sqrt{2}$$

Ex:

$$\int_0^{\pi/4} \tan x\, dx = \int_0^{\pi/4} \frac{\sin x}{\cos x}\, dx$$

- Change of variables: $u = \cos x$

x	0	$\pi/4$
u	1	$\sqrt{2}/2$

- $du = -\sin x\, dx$
- $f(x)\, dx = \frac{\sin x}{u}\, dx = -\frac{du}{u}$
- Integrate:

$$\int_0^{\pi/4} f(x)\, dx = \int_1^{\sqrt{2}/2} -\frac{du}{u} = -\ln|u|\Big|_1^{\sqrt{2}/2} = \ln\left(\frac{\sqrt{2}}{2}\right)$$

* Integration of trigonometric functions:

Ex: How to compute $\int \sin^2 x \, dx$?

Half-angle formula: $\sin^2 x = \dfrac{1-\cos 2x}{2}$

$$\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} \, dx$$

$$= \int \frac{1}{2} \, dx - \int \frac{\cos 2x}{2} \, dx$$

$$= \frac{1}{2}x - \frac{\sin 2x}{4} + C$$

the constant is added at the last step (when all integral signs are removed).

Ex: $\int \sin^3 x \, dx = ?$

One can rewrite the integral as $\int \sin^2 x \underbrace{\sin x \, dx}$

a hint to choose subs. $u = \cos x$

- $u = \cos x$
- $du = -\sin x \, dx$
- $f(x) dx = \sin^2 x \, (-du) = -\sin^2 x \, du = -(1-\cos^2 x) \, du$

$$= -(1-u^2) \, du$$

$$= (u^2-1) \, du$$

- Integrate: $\int f(x) dx = \int (u^2-1) du = \dfrac{u^3}{3} - u + c$

$$= \frac{1}{3}\cos^3 x - \cos x + C$$

In general,

$$\int \sin^m x \cos^n x \, dx$$

If one of m or n is odd, use change of variable
$u = \cos x$ or $u = \sin x$.

If both m and n are even, use the half-angle formula to reduce the power.

Ex:

$$\int \sin^2 x \cos x \, dx \quad \ldots\ldots \quad u = \sin x$$

$$\int \sin^2 x \cos^2 x \, dx \quad \ldots\ldots \quad \text{reduce power: } \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

One gets $\displaystyle\int \frac{1-\cos 2x}{2} \frac{1+\cos 2x}{2} \, dx$

$$= \int \frac{1-\cos^2 2x}{4} \, dx$$

$$= \int \frac{1}{4} \, dx - \frac{1}{4} \int \cos^2 2x \, dx \qquad (*)$$

Reduce power one more time:

$$\cos^2 2x = \frac{1+\cos 4x}{2}$$

$$\int \cos^2 2x = \int \frac{1+\cos 4x}{2} \, dx = \frac{x + \frac{\sin 4x}{4}}{2} + C$$

Plug back to $(*)$:

$$\int \sin^2 x \cos^2 x \, dx = \frac{1}{4} x - \frac{1}{4} \frac{x + \frac{\sin 4x}{4}}{2} + C$$

Ex:

$$\int \sin 2x \cos x \, dx$$

Recall trig-identities:

$$\sin a \cos b = \frac{1}{2}\sin(a+b) - \frac{1}{2}\sin(a-b)$$

$$\cos a \cos b = \frac{1}{2}\cos(a-b) + \frac{1}{2}\cos(a+b)$$

$$\sin a \sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$$

Then $\sin 2x \cos x = \frac{1}{2}\sin 3x - \frac{1}{2}\sin x$.

Thus, $$\int \sin 2x \cos x \, dx = \frac{1}{2}\int \sin 3x \, dx - \frac{1}{2}\int \sin x \, dx$$

$$= \frac{1}{2}\left(-\frac{\cos 3x}{3}\right) - \frac{1}{2}(-\cos x) + C$$

$$= -\frac{\cos 3x}{6} + \frac{\cos x}{2} + C$$