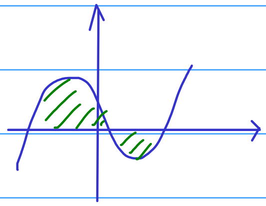
Lecture 7 (1/30/2019)

Recall:



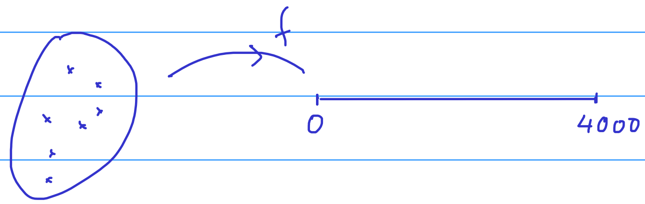
$$\text{Area} = \int_a^b |f(x)|\,dx$$

$$\text{Signed area} = \int_a^b f(x)\,dx$$

$$\text{Length} = \int_a^b \sqrt{1+f'(x)^2}\,dx$$

How about average?

One knows how to compute average of a discreet function. For example,



$f(M)$ = # calories you consume on Monday

$f(T)$ = " Tuesday

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$$\text{Average of calory consumption per day} = \frac{f(M)+f(T)+\dots+f(S)}{7}$$

Suppose you wear a Fitbit watch that tells you how much energy your body spends at each time.

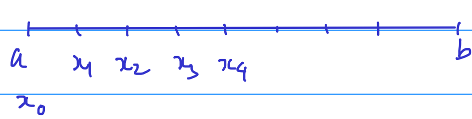
$f(t)$

continuous variable

What is the average energy spent from 8AM to 8:30AM?

* Average of f over interval $[a,b]$:

Recall:



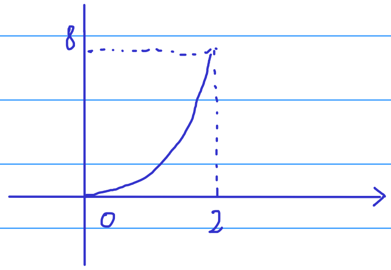
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\,\frac{b-a}{n}$$

Divide both sides by $b-a$:

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=0}^{n-1} f(x_k)\frac{1}{n} = \lim_{n\to\infty}\frac{f(x_0)+f(x_1)+\dots+f(x_{n-1})}{n}$$

This quantity can be reasonably defined as the average of f over the interval $[a,b]$.

Ex: $f(x) = x^3$ on $[0,2]$



$$\text{Avg}(f) = \frac{1}{2-0}\int_0^2 x^3\,dx = \frac{1}{2}\left.\frac{x^4}{4}\right|_0^2 = 2$$

* Velocity, acceleration, displacement:

$$s'(t) = a(t)$$
$$x'(t) = v(t)$$

$s(t)$: position function

$v(t)$: (instant) velocity

$a(t)$: acceleration

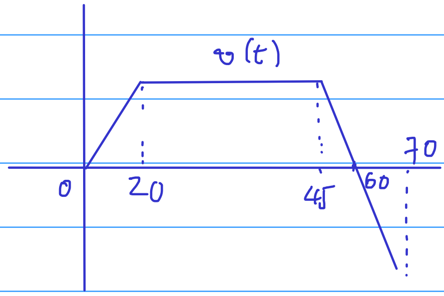
$$s(b)-s(a) = \int_a^b v'(t)\,dt \quad \text{.... displacement between } t=a \text{ and } t=b$$

$$\int_a^b \underbrace{|v(t)|}_{\text{(instant) speed}}\,dt \quad \text{.... distance travelled between } t=a \text{ and } t=b$$

Ex: Suppose you drive with a Snapshot device in your car. At the end of the trip, the device gives you a graph of your speed as a function of time.

Suppose your car's velocity profile is

$$v(t) = \begin{cases} 3t, & 0\le t<20 \\ 60, & 20\le t<45 \\ 240-4t, & t\ge 45 \end{cases}$$



What is the average speed in the first 70 s?

$$\frac{1}{70}\int_0^{70}|v(t)|\,dt = \frac{1}{70}\left(\underbrace{\int_0^{20}|v(t)|\,dt}_{I_1} + \underbrace{\int_{20}^{45}|v(t)|\,dt}_{I_2} + \underbrace{\int_{45}^{70}|v(t)|\,dt}_{I_3}\right)$$

$$I_1 = \int_0^{20} 3t\,dt = \frac{3t^2}{2}\Big|_0^{20} = 600$$

$$I_2 = \int_{20}^{45} 60\,dt = 60t\Big|_{20}^{45} = 1500$$

$$I_3 = \underbrace{\int_{45}^{60}(240-4t)\,dt}_{I_4} + \underbrace{\int_{60}^{70}(4t-240)\,dt}_{I_5}$$

$$\text{Average speed} = \frac{1}{70}(I_1 + I_2 + I_4 + I_5) = \cdots$$

+ Integration by substitution:

Compare differentiation (how to compute derivatives) and integration (how to compute integrals)

Differentiation	Integration
• Sum: $(f+g)' = f' + g'$	$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$
• Scaling: $(cf)' = cf'$ (c constant)	$\int cf(x)\,dx = c\int f(x)\,dx$
• Product: $(fg)' = f'g + fg'$	Integration by parts
• Chain rule: $[f(g(x))]' = f'(g(x))\,g'(x)$	Integration by substitution

Integration by substitution can be summarized in one line:

$$\boxed{\int f(u)\,du = \int f(u(x))\,u'(x)\,dx} \qquad (*)$$

Why is this true?

Let F be an antiderivative of f.

Integrate the equation $[F(g(x))]' = F'(g(x))\, g'(x)$ with respect to x:

$$\int [F(g(x))]' \, dx = \int f(g(x))\, g'(x)\, dx$$

In other words,

$$\int f(g(x))\, g'(x)\, dx = F(g(x)) + C \qquad (**)$$

In practice, this formula amounts to a change of variable

$$\underset{\text{new var.}}{u} = g(\underset{\text{old var.}}{x})$$

Formally, $du = g'(x)\, dx$ (notation from differential calculus)

Thus, one can rewrite $(**)$ as

$$\int f(g(x))\, g'(x)\, dx = F(u) = \int f(u)\, du$$

In practice, the formula $(*)$ is used as

$$\int \underbrace{h(u(x))\, u'(x)}_{f(x)}\, dx = \int h(u)\, du$$

Procedure:

1) Identify a new variable $u = u(x)$ (hidden in $f(x)$)
2) Compute $du = u'(x)\, dx$
3) Change $f(x)\, dx$ into an expression of u only, say $h(u)\, du$
   If this can't be done, the new variable is not a good choice
4) Integrate $\int h(u)\, du$

   The result is a function of u (plus constant C)
5) Substitute back $u = u(x)$.

Ex:

$$\int \underbrace{\frac{x}{x^2+1}}_{f(x)} dx$$

Put $u = x^2+1$. Then $du = u' dx = 2x dx$.

$$f(x)\,dx = \frac{x}{x^2+1}\,dx = \frac{x\,dx}{u} = \frac{\frac{1}{2}du}{u} = \frac{1}{2u}\,du$$

Now integrate both sides:

$$\int f(x)\,dx = \int \frac{1}{2u}\,du = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln(x^2+1) + C$$