Lecture 6 (1/28/2019)

* Arclength: (continued)

Suppose that f is differentiable on (a,b) and continuous on $[a,b]$. Then the Mean Value theorem says that there exists a point x_k^* between x_k and x_{k+1} such that

$$\frac{f(x_{k+1})-f(x_k)}{x_{k+1}-x_k} = f'(x_k^*)$$

Then one can rewrite the sum as

$$\sum_{k=0}^{n-1} \sqrt{1+f'(x_k^*)^2}\,(x_{k+1}-x_k)$$

This is a Riemann sum of $g(x) = \sqrt{1+f'(x)^2}$

Definition:

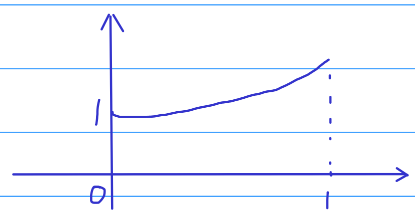
Suppose f is differentiable on (a,b) and continuous on $[a,b]$. Then the length of the graph of f over $[a,b]$ is defined as:

$$l = \int_a^b \sqrt{1+f'(x)^2}\,dx$$

Example:

Length of the curve $y = \frac{1}{2}(e^x+e^{-x})$, $x\in[0,1]$ is

$$L = \int_0^1 \sqrt{1+f'(x)^2}\,dx$$



where $f(x) = \frac{1}{2}(e^x+e^{-x})$

$$f'(x) = \frac{1}{2}(e^x-e^{-x})$$

$$1+f'(x)^2 = 1+\frac{1}{4}(e^{2x}-2\underbrace{e^x e^{-x}}_{1}+e^{-2x})$$

$$= \frac{1}{2}+\frac{1}{4}e^{2x}+\frac{1}{4}e^{-2x}$$

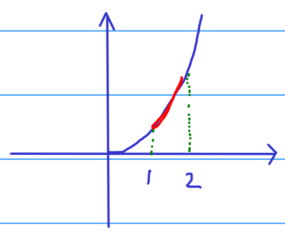
$$= \frac{1}{4}(e^x + e^{-x})^2$$

$$L = \int_0^1 \frac{1}{2}(e^x + e^{-x})\,dx = \frac{1}{2}(e^x - e^{-x})\Big|_0^1 = \frac{1}{2}(e - e^{-1}) - \frac{1}{2}(e^0 - e^0)$$

$$\simeq 1.1752$$

For most functions f, arc length is very hard to compute analytically because it is difficult to find antiderivative of $\sqrt{1+f'(x)^2}$. An approximation method is often used (for example Riemann sum).

Ex: $y = x^2$, $x \in [1,2]$

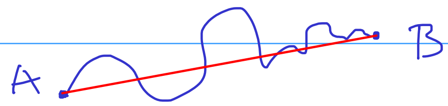


$$f(x) = x^2$$
$$f'(x) = 2x$$
$$f'(x)^2 = 4x^2$$
$$1 + f'(x)^2 = 1 + 4x^2$$

$$L = \int_1^2 \sqrt{1+4x^2}\,dx$$

It's not clear how to eliminate the square root. It's hard to find an antiderivative of $\sqrt{1+4x^2}$. One needs to approximate L by using Riemann sum.

Natural question: Why is the straight path between two points are always the shortest path?



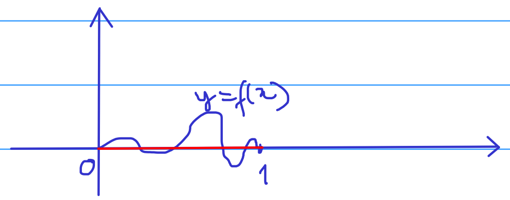
We need the following property: if $f(x) \geq g(x)$ for all $x \in [a,b]$ then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

Why is it true?

$$\begin{cases} \int_a^b f(x)\,dx = \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n-1} f(x_k) \dfrac{b-a}{n} \\ \int_a^b g(x)\,dx = \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n-1} g(x_k) \dfrac{b-a}{n} \end{cases} \quad (\text{and } f(x_k) \geq g(x_k))$$

Choose a coordinate system xy so that A is at $(0,0)$ and B is at $(1,0)$



$$L = \int_0^1 \sqrt{1+f'(x)^2}\,dx \geq \int_0^1 \sqrt{1+0}\,dx$$
$$= 1$$

* Trapezoid rule (or sum)

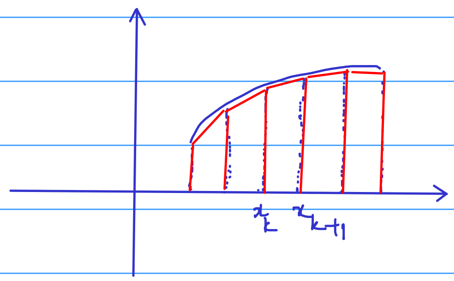
How to compute definite integral $\int_a^b f(x)\,dx$?

- Analytical method: if one can find an antiderivative of $f(x)$, say $F(x)$

  the $\int_a^b f(x)\,dx = F(b) - F(a)$

- Numerical method: approximate the definite integral using Riemann sum
  - left-point R-sum
  - right-point R-sum
  - midpoint R-sum
  - general R-sum

trapezoid sum is a type of general R-sum.



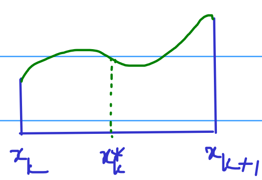
Area of each strip ≈ area of trapezoid slat

$$= \frac{1}{2}\big(\underbrace{f(x_k) + f(x_{k+1})}_{\text{bases}}\big)\underbrace{(x_{k+1} - x_k)}_{\text{height}}$$

$$\text{Area} \approx \sum_{k=0}^{n-1} \frac{f(x_k) + f(x_{k+1})}{2}(x_{k+1} - x_k)$$

Claim: this is a Riemann sum!

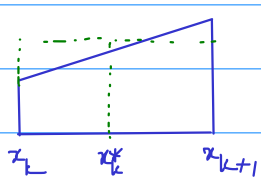
Why? Suppose f is continuous on $[a,b]$.



$\frac{f(x_k) + f(x_{k+1})}{2}$ lies between $f(x_k)$ and $f(x_{k+1})$.

By the Intermediate value theorem, there is x_k^* such that

$$f(x_k^*) = \frac{f(x_k) + f(x_{k+1})}{2}$$



$$\text{Area of trapezoid} = \frac{f(x_k) + f(x_{k+1})}{2}(x_{k+1} - x_k)$$

$$= f(x_k^*)(x_{k+1} - x_k)$$

$\Rightarrow$ Trapezoid sum is a Riemann sum.