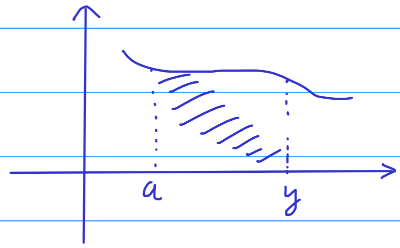
Lecture 5 (1/23/2019)



$$F(y) = \int_a^y f(x)\,dx$$

Recall: $F'(y) = f(y)$

Rename: $F'(x) = f(x)$

F is an antiderivative of f.

Note: $\lim_{\Delta \to 0} \sum_{k=0}^{n-1} f(x_k^*)\Delta x_k$ is denoted by $\int_a^b f(x)\,dx$

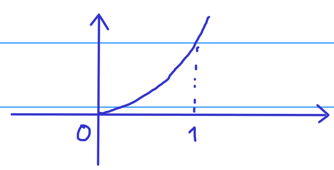
x here is called independent variable. Its name is not important. One can write

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(t)\,dt = \dots$$

Any other antiderivative of f must have the form $G(x) = F(x) + C$ where C is a constant (called constant of integration).

Ex:

Find $\int_0^1 x^2\,dx$



For convenience, we pick a different name for the independent variable under the integral sign: $\int_0^1 t^2\,dt$

Put $F(x) = \int_0^x t^2\,dt$. We know that $F'(x) = x^2$.

Thus, $F(x) = \frac{x^3}{3} + C$. How to find C?

We see that

$$F(0) = \int_0^0 t^2\,dt = 0$$

This implies $C = 0$. Then $F(x) = \frac{x^3}{3}$.

Then $\int_0^1 x^2 dx = F(1) = \frac{1}{3}$.

* Indefinite integral:

The notation $\int f(x)\,dx$ represents all antiderivatives of f.

Ex:

$$\int x\,dx = \frac{x^2}{2} + C$$

$$\int e^x dx = e^x + C$$

$$\int \sin x\,dx = -\cos x + C$$

* Theorem: (Fundamental theorem of calculus)

Let G be an antiderivative of f. Then

$$\int_a^b f(t)dt = G(b) - G(a)$$

Why? $G(x) = F(x) + C$, where $F(x) = \int_a^x f(t)\,dt$

Then $G(b) = F(b) + C$

$G(a) = F(a) + C = C$

Then $G(b) - G(a) = F(b) = \int_a^b f(t)\,dt$.

The value of this theorem can be seen in different aspects:

- Provide an analytical method to compute areas under curves. (When we know an antiderivative of a function, we know the area)
- Connection between two concepts: derivative and integral.

$$\int_a^b G'(t)\,dt = G(b) - G(a)$$

Note that this relation is not trivial from the definition of derivative and integral:

derivative as limit of difference quotient

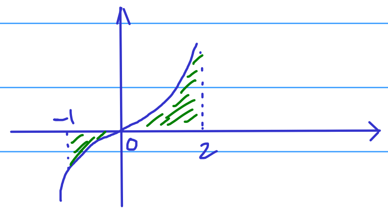
(Definite) integral as limit of Riemann sum.

The difference $G(b) - G(a)$ is often abbreviated as $G(x)\Big|_a^b$

Ex:
$$\int_1^2 x^3\,dx = \frac{x^4}{4}\Big|_1^2 = \frac{2^4}{4} - \frac{1^4}{4} = \frac{15}{4}.$$

$$\int_0^\pi \sin x\,dx = (-\cos x)\Big|_0^\pi = (-\cos\pi) - (-\cos 0) = 1 - (-1) = 2.$$

*Area between curves



The (unsigned) area of the region enclosed between the curves $x=-1$, $x=2$, $y=x^3$, $y=0$ is

$$\int_{-1}^2 |x^3|\,dx = \int_{-1}^0 -x^3\,dx + \int_0^2 x^3\,dx$$

$$= \left(-\frac{x^4}{4}\right)\Big|_{-1}^0 + \frac{x^4}{4}\Big|_0^2$$

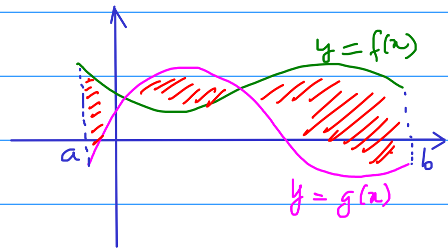
$$= \frac{1^4}{4} + \frac{2^4}{4} = \frac{17}{4}$$

In general, the area of the region between the curves $x=a$, $x=b$, $y=f(x)$, $y=0$ is

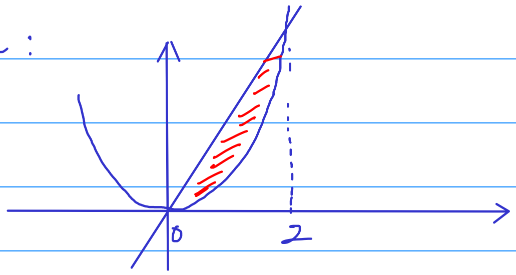
$$\int_a^b |f(x)|\,dx$$

The area of the region between $x=a$, $x=b$, $y=f(x)$, $y=g(x)$ is

$$\int_a^b |f(x) - g(x)|\,dx$$



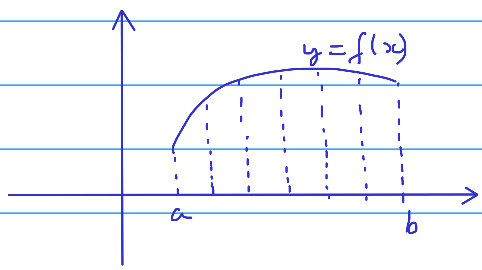
Ex:



Region between $x=0$, $x=2$, $y=x^2$, $y=2x$ has area

$$\int_0^2 (2x-x^2)\,dx = \left(x^2-\frac{x^3}{3}\right)\Big|_0^2 = 2^2-\frac{2^3}{3} = \frac{4}{3}$$

* Length of an arc (revisited):



$$l = \lim_{n\to\infty}\sum_{k=0}^{n-1}\sqrt{(x_{k+1}-x_k)^2+\left(f(x_{k+1})-f(x_k)\right)^2}$$

This sum doesn't look like a Riemann sum. But we claim that it is a Riemann sum of the function $g(x)=\sqrt{1+f'(x)^2}$.

How to see this? Hint:

$$\sum_{k=0}^{n-1}\sqrt{(x_{k+1}-x_k)^2+(f(x_{k+1})-f(x_k))^2} = \sum_{k=0}^{n-1}\sqrt{1+\left(\frac{f(x_{k+1})-f(x_k)}{x_{k+1}-x_k}\right)^2}\,(x_{k+1}-x_k)$$

(continue next time)