Lecture 4 (1/16/2019)

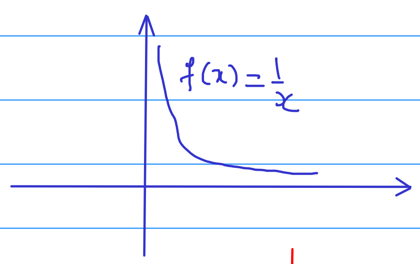
- Trapezoid rule (didn't have a chance to mention last lecture)
- Recall:

  [If $f: [a,b] \to \mathbb{R}$ is bounded and piecewise continuous on (a,b)
  then f is Riemann integrable.]

The Dirichlet's function $f: [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} 1 & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ irrational} \end{cases}$

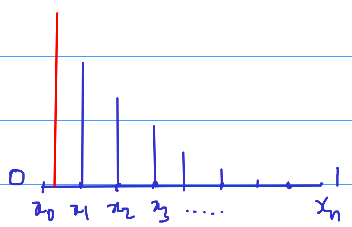
is not Riemann integrable. Note that this function is nowhere continuous. (Why?)

Another example of non-integrable function:



the function $f(x) = \begin{cases} 1/x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$

is piecewise continuous, but not bounded.



Partition the interval $[0,1]$ into n equal subintervals.

$0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$

$x_k = \dfrac{k}{n}$

Take sample point x_k^* on $[x_k, x_{k+1}]$ as follows:

$$x_0^* = \frac{1}{n^2} \qquad \text{(to make } f(x_0^*) \text{ big)}$$

$$x_k^* = x_k = \frac{k}{n} \qquad \text{(left point)}$$

Then the Riemann sum is

$$I_n = f(x_0^*)\frac{1}{n} + f(x_1^*)\frac{1}{n} + f(x_2^*)\frac{1}{n} + \cdots + f(x_{n-1}^*)\frac{1}{n}$$

$$= \frac{1}{x_0^*}\frac{1}{n} + \frac{1}{x_1^*}\frac{1}{n} + \cdots + \frac{1}{x_{n-1}^*}\frac{1}{n}$$

$$= \left(\frac{1}{x_0^*} + \frac{1}{x_1^*} + \cdots + \frac{1}{x_{n-1}^*}\right)\frac{1}{n}$$

$$= \left(\frac{1}{1/n^2} + \frac{1}{1/n} + \frac{1}{2/n} + \cdots + \frac{1}{\frac{n-1}{n}} \right) \frac{1}{n}$$

$$= \left(n^2 + \frac{n}{1} + \frac{n}{2} + \frac{n}{3} + \cdots + \frac{n}{n-1} \right) \frac{1}{n}$$

$$= n + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$$

doesn't converge.

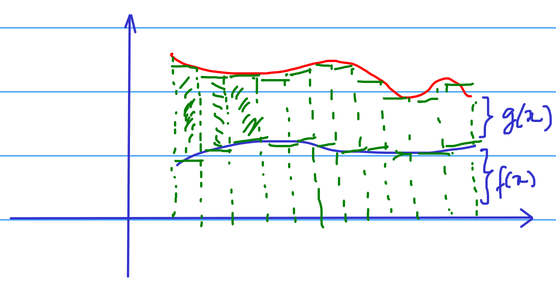
Thm: An unbounded function $f : [a,b] \to \mathbb{R}$ is not R-integrable.

* Recall:

Definite integral is additive:

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

If f and g are integrable on $[a,b]$ then so is $f+g$.



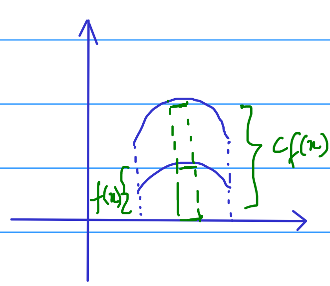
Can be seen from the Riemann sums and from graph.

Definite integral is scaling multiplicative:

$$\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$$

constant

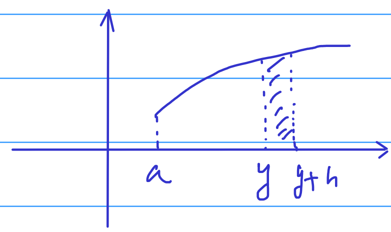
If f is Riemann integrable then so is cf.



$\int_a^b f(x)\,dx$ .... depends only on the function f, and interval $[a,b]$.

How to compute $\int_1^2 x\,dx$ without resorting to Riemann sum?

Put $F(y) = \int_a^y f(x)\,dx$



$F(y+h) - F(y) =$ area of the thin strip

$\approx \underbrace{f(y)}_{\text{height}}\, \underbrace{h}_{\text{width}}$

Approximation gets better as h shrinks to 0.

$$\frac{F(y+h)-F(y)}{h} \approx f(y)$$

In fact, $$\lim_{h\to 0} \frac{F(y+h)-F(y)}{h} = f(y)$$

In other words, $F'(y) = f(y)$

F is an antiderivative of f.

Def: A function $g:(a,b)\to\mathbb{R}$ is called an antiderivative of f if $g'(x) = f(x)$ for all $x\in(a,b)$.

Ex:

$f(x) = x$ has antiderivative $\frac{x^2}{2}$

$f(x) = x^2$ has antiderivative $\frac{x^3}{3}$

$f(x) = x^n$ has antiderivative $\frac{x^{n+1}}{n+1}$ $(n \neq -1)$

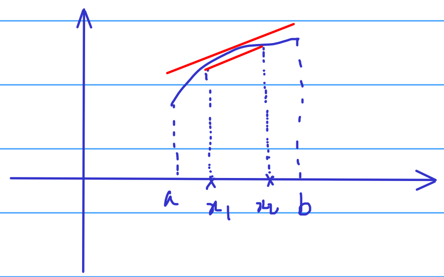
Antiderivative is not unique, for example $\frac{x^2}{2}+1$ is another antiderivative of $f(x) = x$.

Thm: Antiderivatives are unique up to a constant.

Why? Suppose both $g(x)$ and $h(x)$ are antiderivatives of $f(x)$ on (a,b).

$$g'(x) = h'(x) = f(x)$$

Put $k(x) = g(x) - h(x)$. Then $k'(x) = g'(x) - h'(x) = 0$.



We claim that k must be a constant function.

Take two points x_1, x_2 between a and b.

By mean value theorem, there exists $c \in (x_1, x_2)$ such that

$$\frac{k(x_1) - k(x_2)}{x_1 - x_2} = k'(c)$$

this is 0

Therefore, $k(x_1) = k(x_2)$.

Ex :

All antiderivatives of $f(x) = x$ are $\frac{x^2}{2} + C$

" $f(x) = x^2$ " $\frac{x^3}{3} + C$

The notation $\int f(x)dx$ represents all antiderivatives of f.

Ex :

$$\int x\,dx = \frac{x^2}{2} + C$$

called constant of integration

$$\int x^2 dx = \frac{x^3}{3} + C$$

$$\int \cos x\, dx = \sin x + C$$