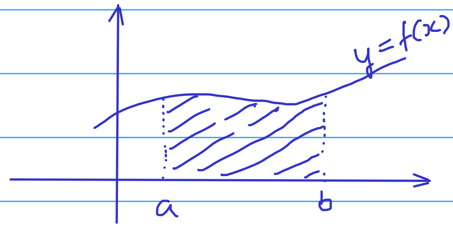
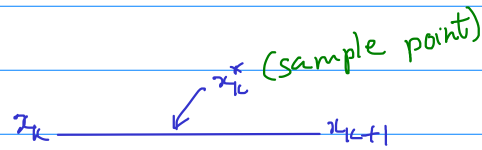
Lecture 3 (1/14/2019)



Partition interval $[a,b]$:

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

$$\text{Area} \approx \sum_{k=0}^{n-1} f(x_k^*)(x_{k+1}-x_k)$$



Left R-sum : $x_k^* = x_k$

Right R-sum : $x_k^* = x_{k+1}$

Midpoint R-sum : $x_k^* = \dfrac{x_k + x_{k+1}}{2}$

General Riemann sum: no restriction on how to choose $x_k^* \in [x_k, x_{k+1}]$.

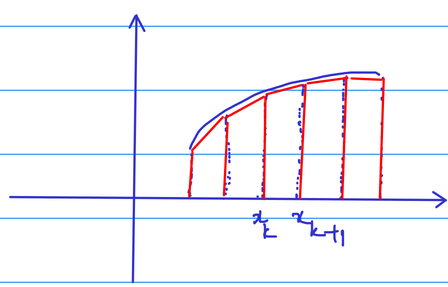
Ex:

$$x_k^* = \frac{2}{3}x_k + \frac{1}{3}x_{k+1}$$

$$x_k^* = \sqrt[3]{x_k^2}\,\sqrt[3]{x_{k+1}}$$

....

* Trapezoid sum:



Area of each strip $\approx$ area of trapezoid slat

$$= \frac{1}{2}\big(f(x_k) + f(x_{k+1})\big)\underbrace{(x_{k+1}-x_k)}_{\text{height}}$$

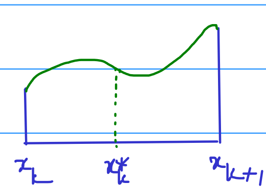
bases

$$\text{Area} \approx \sum_{k=0}^{n-1} \frac{f(x_k)+f(x_{k+1})}{2}(x_{k+1}-x_k)$$

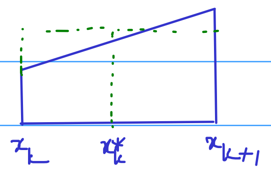
Claim: this is a Riemann sum!

Why? Suppose f is continuous on $[a,b]$.

$\dfrac{f(x_k)+f(x_{k+1})}{2}$ lies between $f(x_k)$ and $f(x_{k+1})$.



By the Intermediate value theorem, there is x_k^* such that $f(x_k^*) = \dfrac{f(x_k)+f(x_{k+1})}{2}$



$$\text{Area of trapezoid} = \frac{f(x_k)+f(x_{k+1})}{2}(x_{k+1}-x_k)$$
$$= f(x_k^*)(x_{k+1}-x_k)$$

$\Rightarrow$ Trapezoid sum is a Riemann sum.

We know how to approximate the area of a region.
Issue: there are many way to approximate. What is the def. of an area?

$$\text{Area} = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k^*)(x_{k+1}-x_k)$$

Will this limit depend on the choice of sample points?
In many cases, no.

Ex:

$$f(x) = x$$

$$S_n = \sum_{k=0}^{n-1} x_k \frac{b-a}{n}, \quad S_n' = \sum_{k=0}^{n-1} x_k^* \frac{b-a}{n}$$

$$S_n - S_n' = \sum_{k=0}^{n-1} \underbrace{(x_k - x_k^*)}_{<0} \frac{b-a}{n}$$

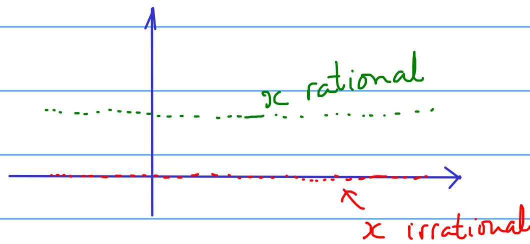
$$|S_n - S_n'| = \sum_{k=0}^{n-1} \underbrace{(x_k^* - x_k)}_{<\frac{b-a}{n}} \frac{b-a}{n} < \sum_{k=0}^{n-1} \frac{b-a}{n}\frac{b-a}{n} = n\frac{(b-a)^2}{n^2}$$
$$= \frac{(b-a)^2}{n} \to 0$$

Thus, S_n and S_n' must have the same limit.

Ex: (pathological function)

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

(called Dirichlet's function)



If we pick x_k^* to be rational, then

$$\sum_{k=0}^{n} \underbrace{f(x_k^*)}_{1} \frac{b-a}{n} = b-a$$

If we pick x_k^* to be irrational then

$$\sum_{k=0}^{n} \underbrace{f(x_k^*)}_{0} \frac{b-a}{n} = 0$$

$$\boxed{\begin{array}{l}\text{\underline{Def}: A function } f \text{ defined on } [a,b] \text{ is said to be \underline{Riemann integrable}} \\ \text{(or simply integrable) if the limit} \\ \qquad \displaystyle\lim_{\Delta \to 0} \sum_{k=0}^{n-1} f(x_k^*)(x_{k+1}-x_k) \\ \qquad (\text{where } \Delta = \max\{x_1-x_0,\, x_2-x_1,\, \dots,\, x_n-x_{n-1}\}) \\ \text{exists and is unique over all partitions of } [a,b] \text{ and all choices} \\ \qquad \text{of sample points } x_k^*.\end{array}}$$

This limit depends only on the function f and the interval $[a,b]$. It is denoted as $\displaystyle\int_a^b f(x)\,dx$, called the <u>(definite) integral of f over $[a,b]$</u>.

About the notation:

$$\int_a^b \underbrace{f(x)}_{\text{integrand}} \underbrace{dx}_{\text{small width}}$$

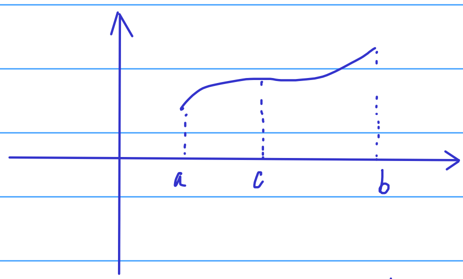
integral sign, looks like letter S (sum)

Integral is a generalization of area. Integral can be negative (depending on whether the integrand is negative).

Integral is signed area: the regions below the x-axis has negative area.

Theorem:

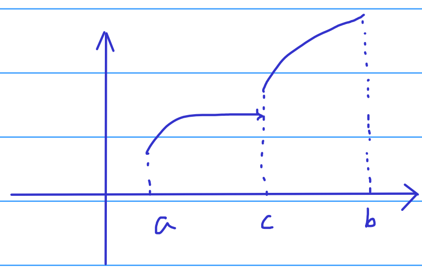
If f is cont. on $[a,b]$ then f is integrable on $[a,b]$.



$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

Convention: $\int_a^a f(x)\,dx = 0$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$



Thm:

If f is piecewise continuous on $[a,b]$ and is bounded on $[a,b]$, then f is Riemann integrable.

The Dirichlet function is not integrable.

Def: $f:(a,b)\to\mathbb{R}$ is said to be piecewise continuous if there exist $c_1, c_2, \cdots, c_k$ (in that order) between a and b such that f is continuous on $(a,c_1),(c_1,c_2),\dots,(c_{k-1},c_k),(c_k,b)$.

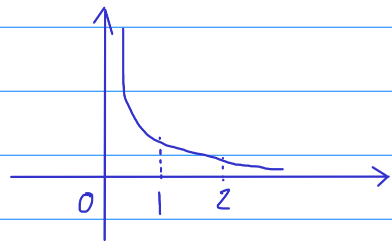
Ex

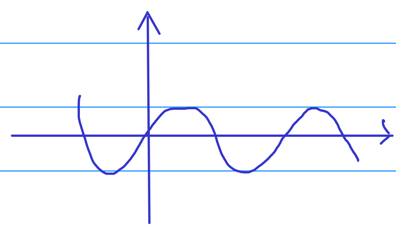
$$f(x) = \begin{cases} \sin x, & x\in(0,1] \\ \dfrac{1}{1+x^2}, & x\in(1,2] \\ e^{-x^2}, & x\in(2,3) \end{cases}$$

is piecewise continuous on $(0,3)$.

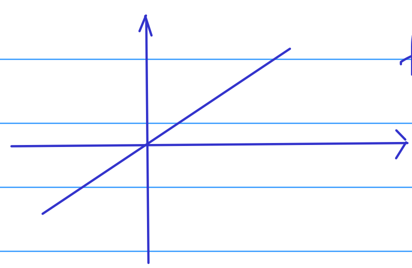
$f:(a,b)\to\mathbb{R}$ is said to be <u>bounded</u> if there exists $M>0$ such that $|f(x)|\le M$ for all $x\in(a,b)$

<u>Ex</u>: $f(x)=\frac{1}{x}$ is bounded on $(1,2)$, but not bounded on $(0,1)$.



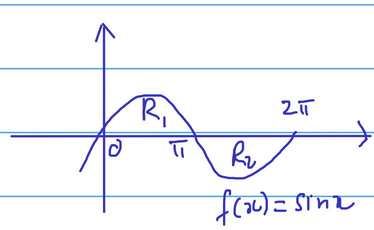


$f(x)=\sin x$ is bounded on $\mathbb{R}$.

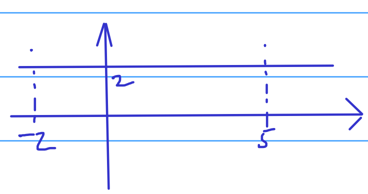


$f(x)=x$ is bounded on $[a,b]$, but not bounded on $\mathbb{R}$.

<u>Theorem</u>: If $f:[a,b]\to\mathbb{R}$ is bounded on $[a,b]$ and piecewise continuous on (a,b) then f is Riemann integrable.



$$\int_0^{2\pi}\sin x\,dx=\text{area}(R_1)-\text{area}(R_2)=0$$



$$\int_{-2}^{5}2\,dx=7(2)=14.$$