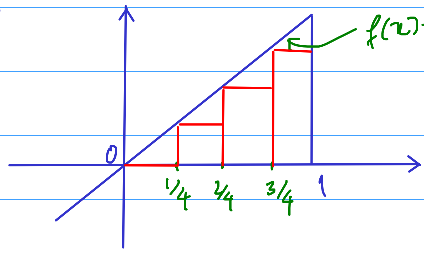
Lecture 2 (1/9/2019)

Examples of computing (approximate) area using Riemann sums:

Ex:



The exact area of the triangle is $\frac{1}{2}$.

The approximate area with

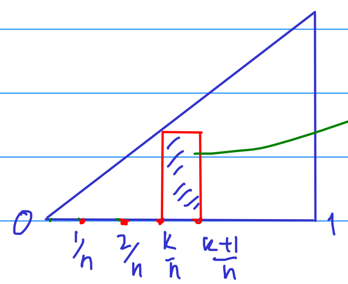
- $n = 4$
- left hand rule

is $0 + \frac{1}{4} \cdot \frac{1}{4} + \frac{2}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{8}$

height width width width

For

- n generic
- left hand rule (i.e using left-point Riemann sum)



area $= \underbrace{\frac{k}{n}}_{\text{height}} \cdot \underbrace{\frac{1}{n}}_{\text{width}} = \frac{k}{n^2}$

Approximate area of triangle $=$ sum of $\frac{k}{n^2}$ for $k = 0, 1, \dots, n-1$

$$= \sum_{k=0}^{n-1} \frac{k}{n^2}$$

How to compute this sum?

$$S_n = \sum_{k=0}^{n-1} \frac{k}{n^2} = \frac{1 + 2 + \cdots + (n-1)}{n^2}$$

$$= \frac{1}{n^2} \sum_{k=0}^{n-1} k$$

Use the identity $(k+1)^2 - k^2 = 2k+1$.

$$
\begin{aligned}
1^2 - 0^2 &= 2\times 0 + 1 \\
2^2 - 1^2 &= 2\times 1 + 1 \\
+ \quad 3^2 - 2^2 &= 2\times 2 + 1 \\
&\cdots\cdots \\
n^2 - (n-1)^2 &= 2\times(n-1) + 1
\end{aligned}
$$

$$\underbrace{\sum_{k=0}^{n-1}\left[(k+1)^2 - k^2\right]}_{n^2 - 0^2} = 2\sum_{k=0}^{n-1} k + n$$

$$n^2 - 0^2 = 2\sum_{k=0}^{n-1} k + n$$

Thus,

$$\sum_{k=0}^{n-1} k = \frac{n^2 - n}{2}$$

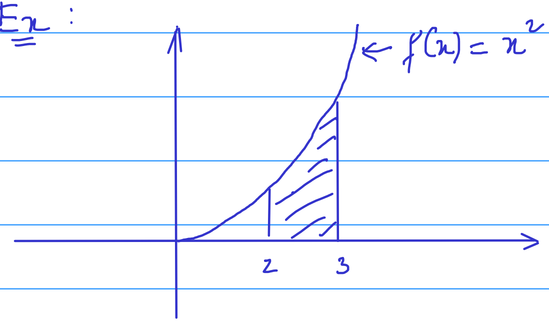
and

$$S_n = \frac{n^2 - n}{2n^2}$$

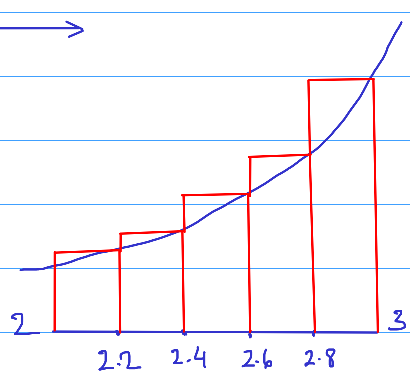
This quantity is the approximate area of the triangle using Riemann sum with n equal subintervals, and left-point rule.

$$S_n \to \underbrace{1/2}_{\text{the exact area}} \quad \text{as} \quad n \to \infty$$

Ex :



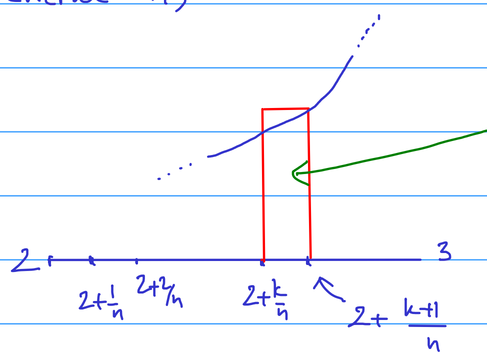
$\begin{cases} n = 2 \\ \text{right-point rule} \end{cases}$



Approximated area $= \underbrace{2.2^2}_{\text{height}} \times \underbrace{0.2}_{\text{width}} + 2.4^2 \times 0.2 + 2.6^2 \times 0.2 + 2.8^2 \times 0.2 + 3^2 \times 0.2$

$= 6.84$

For general n,



$$\text{area} = \underbrace{\left(2+\frac{k+1}{n}\right)^2}_{\text{height}} \times \underbrace{\frac{1}{n}}_{\text{width}}$$

Approximate area of the shade region:

Sum of $\left(2+\frac{k+1}{n}\right)^2 \frac{1}{n}$ over $k = 0, 1, \ldots, n-1$

$$= \sum_{k=0}^{n-1} \left(2+\frac{k+1}{n}\right)^2 \frac{1}{n}$$

$$S_n := \frac{1}{n} \sum_{k=0}^{n-1} \left(2+\frac{k+1}{n}\right)^2$$

How to compute this sum? One can first simplify

$$\sum_{k=0}^{n-1} \left(2+\frac{k+1}{n}\right)^2 = \sum_{j=1}^{n} \left(2+\frac{j}{n}\right)^2 \qquad (j = k+1)$$

which is equal to

$$\sum_{j=1}^{n} \left(4 + 4\frac{j}{n} + \frac{j^2}{n^2}\right)$$

which is equal to

$$\underbrace{\sum_{j=1}^{n} 4}_{4n} + \underbrace{\sum_{j=1}^{n} 4\frac{j}{n}}_{\frac{4}{n}\sum_{j=1}^{n} j} + \underbrace{\sum_{j=1}^{n} \frac{j^2}{n^2}}_{\frac{1}{n^2}\sum_{j=1}^{n} j^2}$$

We know that $\sum_{j=1}^{n} j = 1+2+\cdots+n = \frac{(n+1)^2-(n+1)}{2} = \frac{n^2+n}{2}$

How to find $\sum_{j=1}^{n} j^2$?

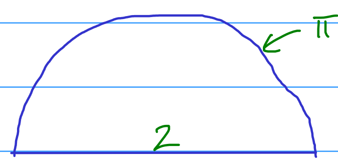
Expand the expression $(j+1)^3 - j^3$ then sum up over $j=1,2,..,n$

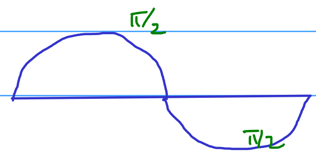
(Homework)

After all, $S_n \to 19/3$ as $n \to \infty$.

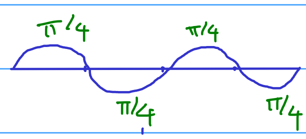
* Length of arcs:

"Length paradox":



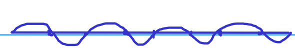


length of arc = π, length of seg. = 2



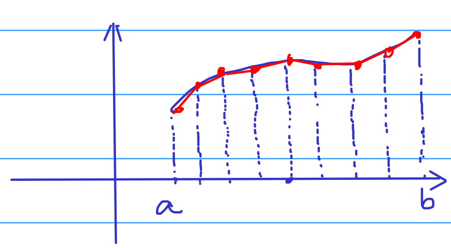
" = π, length of seg. = 2

⋮

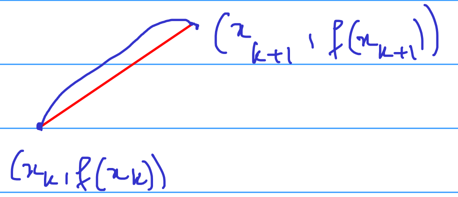


" = π, length of seg. = 2

How to define length of an arc?



use Archimedes-Riemann's idea: partition interval $[a,b]$ into n equal subintervals by points $a = x_0 < x_1 < \cdots < x_n = b$



The length of the arc between two consecutive points is approximated by the length of the segment:

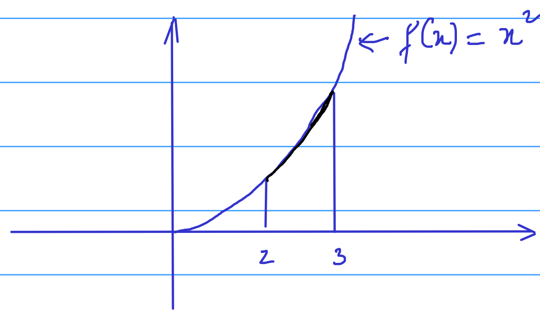
$$d_k = \sqrt{(x_{k+1} - x_k)^2 + (f(x_{k+1}) - f(x_k))^2}$$

Then the length of the whole arc is approximated by

$$\sum_{k=0}^{n-1} d_k = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + (f(x_{k+1}) - f(x_k))^2}$$

Note: $x_{k+1} - x_k = \frac{b-a}{n}$, the width of each subinterval.

Ex:



$x_0 = 2$

$x_1 = 2 + \frac{1}{n}$

$x_2 = 2 + \frac{2}{n}$

...

$x_k = 2 + \frac{k}{n}$

...

$x_n = 2 + \frac{n}{n} = 3$

$$\text{length} \approx \sum_{k=0}^{n-1} \sqrt{\frac{1}{n^2} + \left[\left(2 + \frac{k+1}{n}\right)^2 - \left(2 + \frac{k}{n}\right)^2\right]^2}$$

For large n, say $n = 100$, one needs Mathematica to compute.