Lecture 18 (3/13/2019)

* Logistic model:

$$y'(t) = k(y-a)(y-b)$$

where a, b, k are constants.

How to solve?

$$\frac{dy}{dt} = k(y-a)(y-b)$$

$$\Rightarrow \frac{dy}{(y-a)(y-b)} = k\,dt \qquad \text{(separation of variables)}$$

$$\underbrace{\int \frac{dy}{(y-a)(y-b)}}_{\text{use partial fractions}} = \int k\,dt$$

- Problem: A population of rabbits grows in time $P(t)$. At the initial time, there are 8 rabbits. Due to food and space constraint, the population doesn't exceed 100.

Observation:

$$\frac{dP}{dt} \approx 0 \quad \text{when } t \approx 0 \quad \text{(due to small population)}$$

$$\frac{dP}{dt} \text{ is large when } P \text{ is somewhere near } 50$$

$$\frac{dP}{dt} \approx 0 \quad \text{when } P \approx 100$$

A simple model that captures these features is

$$\frac{dP}{dt} = \underbrace{k}_{\text{some constant}} P(100-P)$$

Suppose $k=2$. How to solve for $P(t)$?

$$\frac{dP}{P(100-P)} = 2\,dt$$

Partial fractions:

$$\frac{1}{P(100-P)} = \frac{1}{100}\left(\frac{1}{P} + \frac{1}{100-P}\right)$$

Integrate:

$$\int \frac{1}{100}\left(\frac{1}{P} + \frac{1}{100-P}\right) dP = \int k\,dt$$

or

$$\frac{1}{100} \ln \frac{P}{100-P} = 2t + C$$

or

$$\ln \frac{P}{100-P} = 200t + \underbrace{100C}_{C_1}$$

or

$$\frac{P}{100-P} = e^{200t + C_1} = e^{200t} \underbrace{e^{C_1}}_{C_2} = C_2 e^{200t}$$

Plug $t=0$ to find C_2:

$$\frac{8}{100-8} = C_2 e^0 \rightsquigarrow C_1 = \frac{8}{92} = \frac{2}{23}$$

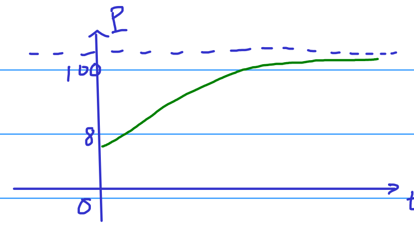
Substitute back:

$$\frac{P}{100-P} = \frac{2}{23} e^{200t}$$

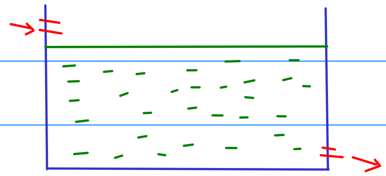
$$\Rightarrow \quad P = \frac{2}{23} e^{200t}(100-P) = \frac{200}{23} e^{200t} - \frac{2}{23} e^{200t} P$$

$$\Rightarrow \quad \left(1 + \frac{2}{23} e^{200t}\right) P = \frac{200}{23} e^{200t}$$

$$\Rightarrow \quad P(t) = \frac{\frac{200}{23} e^{200t}}{1 + \frac{2}{23} e^{200t}}$$



* Mixing:



A tank contains $V = 5000$ gal of brine. The amount of salt at the initial time is $s(0) = 15$ (lbs). People try to purify the water in the tank by pouring fresh water at the rate of 5 gal/min while draining the tank at the same rate (in order to keep the volume the same at all time). Find the amount of salt (lbs) as a function of time.

$$\underbrace{\frac{ds}{dt}}_{\substack{\text{net rate} \\ \text{of change} \\ \text{lbs/min}}} = \underset{\text{lbs}}{(\text{rate salt in})} - \underset{\text{lbs}}{(\text{rate salt out})}$$

$$= \underset{\text{gal/min}}{(\text{rate flow in})}\underset{\text{lbs/gal}}{(\text{concentration in})}$$

$$- \underset{\text{gal/min}}{(\text{rate flow out})}\underset{\text{lbs/gal}}{(\text{concentration out})}$$

$$= 5(0) - 5\frac{s}{5000} = \frac{-s}{1000}$$

$$\frac{ds}{dt} = -\frac{s}{1000} \Rightarrow \frac{ds}{s} = -\frac{1}{1000}dt$$

Integrate: $\quad \ln s = -\frac{1}{1000}t + C$

Exponentiate: $\quad s = e^{-\frac{1}{1000}t} e^{C} = k\, e^{-\frac{t}{1000}}$

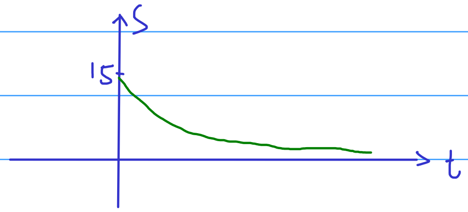
To find k, plug $t=0$:

$$15 = s(0) = k e^{0} = k$$

Conclution:

$$s(t) = 15\, e^{-\frac{t}{1000}}$$

The water is purified slowly at first, and more rapidly later.



Now with the same hypotheses as above, but assuming the volume rate of the inward flow is 4 gal/min.

$$V(t) = 5000 - t \quad (gal)$$

$$\frac{ds}{dt} = 4(0) - 5\,\frac{s}{5000-t} = -\frac{5s}{5000-t}$$

Separate s and t:

$$\frac{ds}{s} = -\frac{5\,dt}{5000-t}$$

Integrate:
$$\int \frac{ds}{s} = -\int \frac{5\,dt}{5000-t}$$

*Practice problems:

1) Find $y = y(t)$:

(a) $t^2 \dfrac{dy}{dt} = \sqrt{y}\,(2t+1)$, $\quad y(1) = \frac{1}{2}$

(b) $\dfrac{dy}{dt} = 2(y-1)(y-3)$, $\quad y(0) = 2$

2)

3) on worksheet for next class

4)