# Lecture 17 (3/11/2019)

Differential equations of order one:
(or first order differential equations)

$$y' = f(x,y) \quad \text{---- a relation between } x, y, y',$$

where $y$ is a function of $x$.

Ex:

$$y' = x + y,$$
$$y' = ye^{x},$$
$$y' = \sin(xy),$$
$$\ldots$$

Such differential equation is found in various mathematical models:

1) Compound interest:

Suppose one deposites an amount of \$1000 in his saving account. The interest is compounded continuously with rate 0.2% (annual percentage yield). How much the balance have in 1 year?

The answer is not $1000(1+0.002)$ because the interest is accumulated continuously over the year, not just at the end of the year.

$y = y(t)$: balance at time $t$ (year)

$y(0) = 1000$

Equation:
$$\underbrace{\frac{dy}{dt}}_{\text{rate of increase at time } t} = (1+0.002)\underbrace{y}_{\text{balance at time } t} = 1.002\, y$$

Solve:
$$\frac{dy}{y} = 1.002\, dt$$

Integrate both sides:

$$\ln y = 1.002t + C$$

Exponentiate both sides:

$$y = e^{1.002t+C} = k\, e^{1.002t}$$

For $t=0$: $\quad 1000 = k e^0 = k$

Thus,

$$y(t) = 1000\, e^{1.002t}$$

Answer: $\quad y(1) = 1000\, e^{1.002} \approx 2723.72$

2) <u>Newton's law of cooling</u>

$T_0$: temperature of the medium

temperature (of the rod at time $t$)

$T(t)$

Observations:

- If $T(t) > T_0$ then $T'(t) < 0$.
- If $T(t) < T_0$ then $T'(t) > 0$.

$\Big\}$ quite intuitive

It seems $T(t) - T_0 \sim T'(t)$ (proprotional)

Newton's law of cooling:

$$T'(t) = -\underset{\uparrow}{k}\,(T - T_0)$$

heat constant $>0$

<u>Ex</u>:

$k=2$, $T_0 = 72$, $T(0) = 100$ (initial temperature)

Equation:

$$T' = \frac{dT}{dt} = -2(T-72)$$

How to solve for $T$ as a function of time?

First separate the variables $T$ and $t$:

$$\frac{dT}{T-72} = -2dt$$

Integrate both sides:

$$\int \frac{dT}{T-72} = \int -2dt$$

$$\ln|T-72| = -2t + C$$

Note: $T > 72$ at all time because the initial temperature on the rod is 100, greater than 72 (the temperature of the medium).

$$|T-72| = T-72$$

Exponentiate:

$$T-72 = e^{-2t+C} = e^{-2t}\underbrace{e^{C}}_{C_1}$$

$$\Rightarrow \quad T = 72 + C_1 e^{-2t}$$

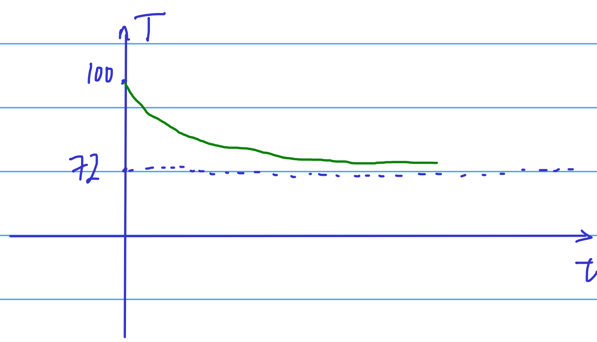
Plug $t=0$:

$$100 = 72 + C_1 e^0 = 72 + C_1$$

$$C_1 = 28$$

Conclusion:

$$T(t) = 72 + 28e^{-2t}$$




More models will be discussed next time.

* Solve differential equations by separation of variable method:

$$\frac{dy}{dt} = \underbrace{f(y)\,g(t)}_{\text{separable form}}$$

(Non-separable form: $y+t$, $\sin(yt)$, ...)

Strategy: split $y$ and $t$ by writing

$$\frac{dy}{f(y)} = g(t)\,dt$$

Then integrate both sides: $\displaystyle\int \frac{dy}{f(y)} = \int g(t)\,dt$

Ex:

$$\frac{dy}{dt} = yt, \qquad y(0) = 1$$

Separate $y$ from $t$:

$$\frac{dy}{y} = t\,dt$$

Integrate: $\ln y = \frac{t^2}{2} + C$

Exponentiate: $y = e^{\frac{t^2}{2}+C} = k e^{\frac{t^2}{2}}$

Substitute $t=0$ to find $k$:

$$1 = k e^0$$

Thus, $k=1$.

Conclusion: $y(t) = e^{t^2/2}$.

More examples on worksheet.