Lecture 16 (3/6/2019)

* Improper integrals:

Recall: the definite integral $\int_a^b f(x)\,dx$

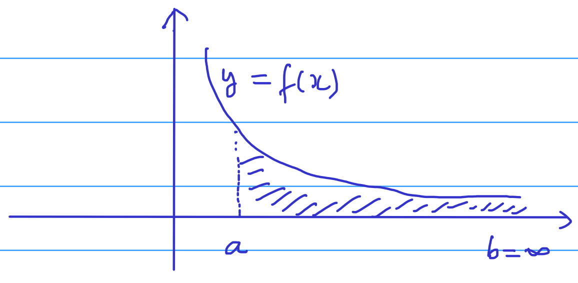
is defined if f is Riemann integrable on $[a,b]$.

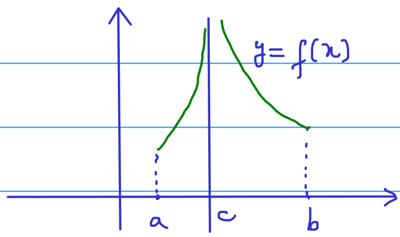
- The interval $[a,b]$ has to be finite.
- If f is R-integrable then f is bounded on $[a,b]$, that is

$$|f(x)| \le M \quad \text{for all } x \in [a,b]$$

for some $M > 0$.

[An "integral" of the form $\int_a^b f(x)\,dx$ is said to be improper integral if $a = \pm\infty$ or $b = \pm\infty$ or f is unbounded on (a,b).]





Interval (a,b) is infinite

f is unbounded

<u>Note</u>: improper integral is not Riemann integral, but is a reasonable extension of Riemann integral.

How to define $\int_a^{\infty} f(x)\,dx$ or $\int_{-\infty}^{b} f(x)\,dx$?

The most natural way is to define:

$$\int_a^{\infty} f(x)\,dx \overset{\text{def}}{=\!=} \lim_{c\to\infty} \int_a^c f(x)\,dx$$

$$\int_{-\infty}^{b} f(x)\,dx \overset{\text{def}}{=\!=} \lim_{c\to-\infty} \int_c^b f(x)\,dx$$

If the limit exists and finite, the improper int. is said to <u>converge</u>.

Otherwise, it is said to <u>diverge</u>.

Ex:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \underbrace{\int_1^b \frac{1}{x^2}\,dx}$$

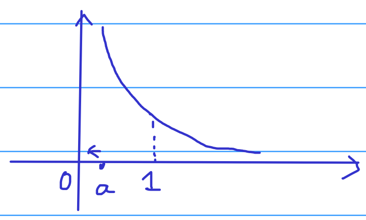
$$= -\frac{1}{x}\Big|_1^b = 1-\frac{1}{b} \longrightarrow 1 \text{ as } b\to\infty$$

Thus, the improper int. $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges and

$$\int_1^{\infty} \frac{1}{x^2}\,dx = 1.$$

Ex:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

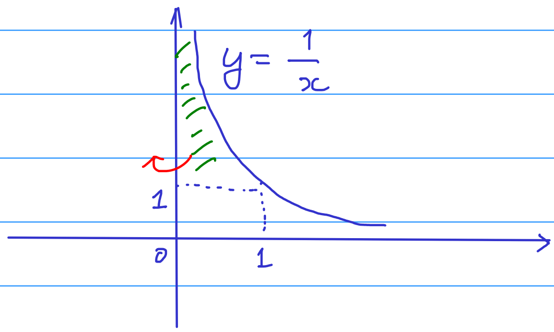


The integrand is not bounded on the interval $(0,1)$, but is bounded on interval $(a,1)$ for any $0<a<1$.

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \underbrace{\int_a^1 \frac{1}{\sqrt{x}}\,dx}_{2\sqrt{x}\big|_a^1 \,=\, 2-2\sqrt{a}}$$

The limit exists and is equal to 2. Therefore, the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ exists and is equal to 2.

Ex:

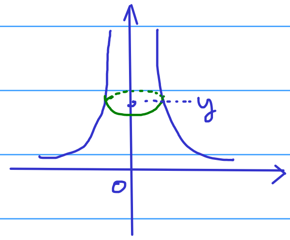


Consider an infinite "tent" obtained by revolve the curve

$$y = \frac{1}{x}, \quad x \in (0,1]$$

about the y-axis. What are the volume and surface area of the tent?

- Volume: use slice method: slice area at position y is



$$\pi x^2 = \frac{\pi}{y^2}$$

$$\text{vol} = \int_1^\infty \frac{\pi}{y^2}\,dy = \pi$$

- Surface area:

$$f(y) = \frac{1}{y}$$

$$S = \int_1^\infty 2\pi f(y)\sqrt{1+f'(y)^2}\,dy \geq \int_1^\infty 2\pi f(y)\,dy = \int_1^\infty \frac{2\pi}{y}\,dy$$

Observe that $\int_1^\infty \frac{dy}{y} = \lim_{b\to\infty} \int_1^b \frac{dy}{y} = \lim_{b\to\infty} \ln b = \infty$

Therefore $S=\infty$. This tent has a strange property: there is enough paint to paint the inward, but not enough to paint the outward.