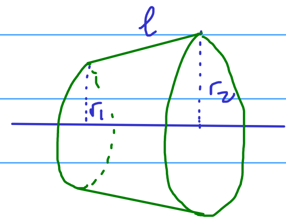
Lecture 14 (3/1/2019)

Area of surface of rotation:



frustum of a cone

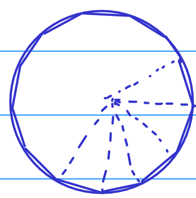
r_1 ... radius of smaller base

r_2 ... radius of larger base

ℓ .... slant height of the frustum

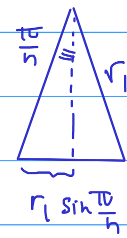
Question: what is the (latteral) area of the frustum in terms of r_1, r_2, ℓ?

Divide each base into n equal sectors. We obtain a regular n-polygon. The side of this polyon is of length



$$a = 2 r_1 \sin \frac{\pi}{n} \qquad \text{(for smaller base)}$$

$$b = 2 r_2 \sin \frac{\pi}{n} \qquad \text{(for larger base)}$$



The frustum is approximated by n slats of trapezoid shape.

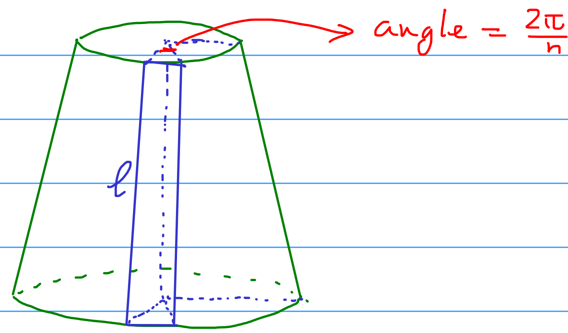
Each of bases a, b,

height ℓ

The area of each slat is

$$\frac{a+b}{2}\ell$$

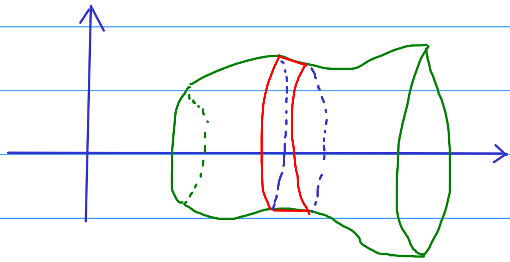
$$= \ell (r_1 + r_2) \sin \frac{\pi}{n}$$



The area of the frustum is approximately

$$n \ell (r_1 + r_2) \sin \frac{\pi}{n} = \ell (r_1 + r_2) \underbrace{n \sin \frac{\pi}{n}}_{\to \pi \ (\text{as } n \to \infty)} \longrightarrow \ell (r_1 + r_2) \pi$$

Therefore, the area of the frustum (of the cone) is $(r_1+r_2)\ell\pi$.



Area of frustum

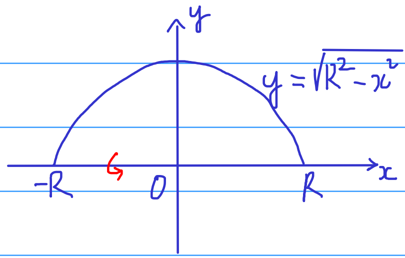
$$\approx \pi \left(\underbrace{f(x_k)}_{\text{base radius}} + \underbrace{f(x_{k+1})}_{\text{base radius}}\right) \underbrace{\sqrt{1+f'(x_k)^2}\, h}_{\text{slant height}}$$

$$\approx 2\pi f(x_k) \sqrt{1+f'(x_k)^2}\, h$$

Area of surface of revolution $\approx \sum_k 2\pi f(x_k)\sqrt{1+f'(x_k)^2}\, h$

Exact area $= \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\, dx$

Ex: What is the surface area of the sphere of radius R?



$$f(x) = \sqrt{R^2-x^2}$$

$$f'(x) = \frac{-x}{\sqrt{R^2-x^2}}$$

$$f'(x)^2 = \frac{x^2}{R^2-x^2}$$

$$1+f'(x)^2 = 1+\frac{x^2}{R^2-x^2} = \frac{R^2}{R^2-x^2}$$

$$\sqrt{1+f'(x)^2} = \frac{R}{\sqrt{R^2-x^2}}$$

$$2\pi f(x)\sqrt{1+f'(x)^2} = 2\pi\sqrt{R^2-x^2}\,\frac{R}{\sqrt{R^2-x^2}} = 2\pi R$$

Area of sphere $= \int_{-R}^{R} 2\pi R \, dx = 2\pi R x \Big|_{-R}^{R} = 4\pi R^2$