Lecture 13 (2/27/2019)

* Integration by parts :
  - Indefinite integral :

$$\int u \underbrace{v' dx}_{dv} = uv - \int v \underbrace{u' dx}_{du}$$

  - Definite integral :

$$\int_a^b u v' dx = uv \Big|_a^b - \int_a^b v u' dx$$

Ex : $I = \int_0^{\pi} x^2 \cos x \, dx$

$u = x^2$ .... $du = 2x\,dx$

$dv = \cos x \, dx$ ..... $v = \sin x$

$$I = \underbrace{x^2 \sin x \Big|_0^{\pi}}_{=0} - \int_0^{\pi} 2x \sin x \, dx = \int_0^{\pi} -2x \sin x \, dx$$

$u = -2x$ .... $du = -2dx$

$dv = \sin x \, dx$ .... $v = -\cos x$

$$I = (-2x)(-\cos x) \Big|_0^{\pi} - \int_0^{\pi} (-\cos x)(-2\,dx)$$

$$= 2x \cos x \Big|_0^{\pi} - \int_0^{\pi} 2 \cos x \, dx$$

$$= 2\pi \cos \pi - 2(0) \cos 0 - 2 \sin x \Big|_0^{\pi}$$

$$= -2\pi$$

Ex: $I = \int e^x \sin x \, dx$

$u = e^x \quad \ldots \quad du = e^x dx$

$dv = \sin x dx \quad \ldots\ldots \quad v = -\cos x$

$$I = -e^x \cos x + \int e^x \cos x \, dx$$

$u = e^x \quad \ldots\ldots \quad du = e^x dx$

$dv = \cos x dx \quad \ldots\ldots \quad v = \sin x$

$$I = -e^x \cos x + e^x \sin x - \underbrace{\int e^x \sin x \, dx}_{I}$$

Therefore, $I = \dfrac{-e^x \cos x + e^x \sin x}{2}$

More examples on worksheet

* Partial fraction decomposition:

How to integrate rational function $f(x) = \dfrac{P(x)}{Q(x)}$? polynomial

$$\frac{x+1}{x^2+x-2}$$

Note: $x^2+x-2 = (x-1)(x+2)$

Write $\dfrac{x+1}{x^2+x-2} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$

Equate numerators: $x+1 = \underbrace{A(x+2) + B(x-1)}_{(A+B)x + 2A-B}$

We get a system of two equations and two unknowns:

$$\begin{cases} A+B = 1 \\ 2A-B = 1 \end{cases}$$

From the first eq. $B = 1-A$

Substitute B into second eq. $\underbrace{2A-(1-A)}_{3A-1} = 1$

This gives $A = 2/3$.

Then $B = 1 - A = 1/3$

$$\frac{x+1}{x^2+x-2} = \frac{1/3}{x-1} + \frac{2/3}{x+2}$$

$$\int \cdots\, dx = \int \frac{1/3}{x-1}\,dx + \int \frac{2/3}{x+2}\,dx$$

$$= \frac{1}{3}\ln|x-1| + \frac{2}{3}\ln|x+2| + C$$

Ex:

$$\frac{x^3+x+1}{x^2+x-2}$$

Degree of numerator is not less than degree of denominator. We need to do long division first.

$$\begin{array}{r l}
 & x-1 \quad \leftarrow \text{quotient} \\
x^2+x-2 & \overline{)\,x^3+0+x+1} \\
- & x^3+x^2-2x \\
\hline
 & 0-x^2+3x+1 \\
- & -x^2-x+2 \\
\hline
 & 0+4x-1 \quad \leftarrow \text{remainder}
\end{array}$$

$$\frac{x^3+x+1}{x^2+x-2} = x-1 + \frac{4x-1}{x^2+x-2}$$

Partial fraction:

$$\frac{4x-1}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2}$$

Equate numerators:

$$4x-1 = A(x+2) + B(x-1)$$

Plug $x=1$: $\quad 3 = 3A \rightsquigarrow A = 1$

Plug $x=-2$: $\quad -9 = -3B \rightsquigarrow B = 3$

$$\frac{x^3+x+1}{x^2+x-2} = x-1 + \frac{1}{x-1} + \frac{3}{x+2}$$

$$\int \cdots dx = \int (x-1)\,dx + \int \frac{dx}{x-1} + \int \frac{3}{x+2}$$

$$= \frac{x^2}{2} - x + \ln|x-1| + 3\ln|x+2| + C$$

Ex

$$\frac{x^2-x+1}{x^3-x^2} = \frac{x^2-x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

Equate numerators:

$$x^2-x+1 = Ax(x-1) + B(x-1) + Cx^2$$

Plug $x=1$: $\quad 1 = C$

Plug $x=0$: $\quad 1 = -B \rightsquigarrow B = -1$

Substitute back:

$$x^2-x+1 = Ax(x-1) - (x-1) + x^2$$

$$\rightsquigarrow \underbrace{x^2-x+1 + (x-1) - x^2}_{=0} = Ax(x-1)$$

$$\rightsquigarrow A = 0$$

$$\frac{x^2-x+1}{x^3-x^2} = -\frac{1}{x^2} + \frac{1}{x-1}$$

$$\int \cdots dx = \int -\frac{dx}{x^2} + \int \frac{dx}{x-1} = \frac{1}{x} + \ln|x-1| + C$$