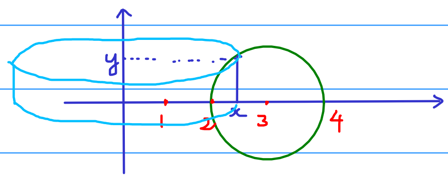
Lecture 12 (2/18/2019)

Volume of torus:



shell method:

chopping shells along the x-axis

- base radius $= x$
- height $= y = \sqrt{1-(x-3)^2}$
- lateral area of cylinder:

$$2\pi x\sqrt{1-(x-3)^2}$$

Equation of circle:

$$(x-3)^2 + (y-0)^2 = 1$$

On the upper half: $y \geq 0$

$$y = \sqrt{1-(x-3)^2}$$

$$\text{volume} = 2\int_2^4 2\pi x\sqrt{1-(x-3)^2}\,dx$$

How to compute this integral?

$u = 1-(x-3)^2$ won't work

Try: $u = x-3 \Rightarrow \begin{cases} du = dx \\ x = u+3 \end{cases}$

$$I = \int_2^4 2\pi x\sqrt{1-(x-3)^2}\,dx = \int_{-1}^{1} 2\pi(u+3)\sqrt{1-u^2}\,du$$

$$= 2\pi \underbrace{\int_{-1}^{1} u\sqrt{1-u^2}\,du}_{I_1} + 6\pi \underbrace{\int_{-1}^{1}\sqrt{1-u^2}\,du}_{I_2}$$

* Compute I_1:

$v = 1-u^2$

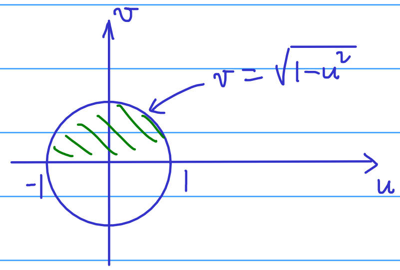
$dv = -2u\,du$

u	-1	1
v	0	0

$$I_1 = \int_0^0 -\frac{1}{2}\sqrt{v}\,dv = 0$$

* Compute I_2:

Two ways: 1) Use geometry



$I_2 =$ area of semicircle

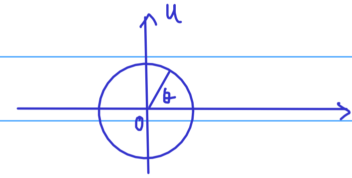
$= \frac{1}{2}(\pi 1^2) = \frac{\pi}{2}$

2) Use trigonometry substution:

Put $u = \sin\theta$ (i.e. $\theta = \arcsin u$)

$du = \cos\theta \, d\theta$

u	-1	1
θ	$-\pi/2$	$\pi/2$



$\sqrt{1-u^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$ (because $\cos\theta \geq 0$)

$$I_2 = \int_{-\pi/2}^{\pi/2} \cos\theta \cos\theta \, d\theta = \int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \left. \frac{\theta + \frac{1}{2}\sin 2\theta}{2} \right|_{-\pi/2}^{\pi/2}$$

$$= \frac{\pi}{2}$$

* In general, how to evaluate the integral

$$\int \sqrt{1-x^2} \, dx \quad ?$$

Put $x = \sin\theta$, or $\theta = \arcsin x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$dx = \cos\theta \, d\theta$

$$\sqrt{1-x^2} \, dx = \sqrt{1-\sin^2\theta} \cos\theta \, d\theta = \cos^2\theta \, d\theta = \frac{1+\cos 2\theta}{2} d\theta$$

Integrate both sides:

$$\int \sqrt{1-x^2}\,dx = \int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{\theta + \frac{\sin 2\theta}{2}}{2} + C$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$$

$$= \frac{1}{2}\arcsin x + \frac{1}{4}\sin(2\arcsin x) + C$$

Ex:

$$I = \int_0^1 \sqrt{9-4x^2}\,dx \qquad ?$$

Put $u = \frac{2}{3}x \quad \left(\Rightarrow du = \frac{2}{3}dx\right)$

$$I = \int_0^{2/3} \sqrt{9-9u^2}\,\frac{3}{2}\,du = \frac{9}{4}\int_0^{2/3} \sqrt{1-u^2}\,du$$

<u>Integration by parts</u>

Recall: integration by substitution $\longleftrightarrow$ chain rule

$$[h(u(x))]' = h'(u(x))\,u'(x)$$

$$\int \underbrace{h'(u(x))\,u'(x)}_{f(x)}\,dx = h(u(x)) + C$$

integration by parts $\longleftrightarrow$ product rule (Leibniz's rule)

$$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x)$$

Integrate:

$$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx$$

or

$$\int u(x)\underbrace{v'(x)dx}_{dv} = u(x)v(x) - \int v(x)\underbrace{u'(x)dx}_{du}$$

Integration by parts

Briefly, $\int u\,dv = uv - \int v\,du$

Ex:

$$\int x e^x dx$$

$u = x \qquad\qquad du = dx$

$dv = e^x dx \longrightarrow v = e^x$

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C$$