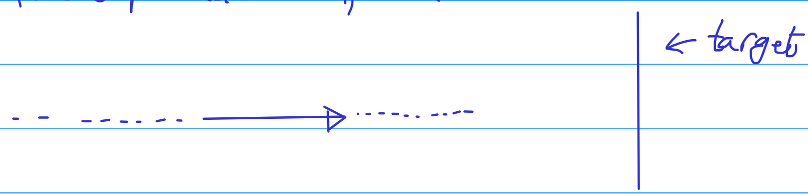
Lecture 1 (1/7/2019)

* Some paradoxes:

1) Arrow paradox (of Zeno):



At each instant of time, the arrow is at rest. Thus, the arrow is always at rest and cannot approach the target (!)

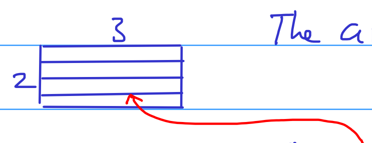
2) "Length" paradox:

0 ——— 1 The interval $[0,1]$ has length 1.

It consists of points. Each point is an interval of length 0.

Claim: The length of the interval is equal to the sum of the lengths of these points, which is 0 (!)

3) "Area" paradox:



The area of a rectangle of length 3 and width 2 is $2 \times 3 = 6$.

A segment can be thought as a rectangle of length 3 and width 0.

The rectangle is the union of those segments. Thus, its area is 0 (!)

In both cases, there's an issue with "summing".

The common fallacy in both paradoxes is the idea of "quanty over an interval = sum of quantity over each point of that interval."

An interval is made of $\underbrace{\text{"too many"}}_{\text{uncountably infinite}}$ points. It is illegitimate to perform such a sum.

One can sum only countably (either finite or inf.) many quantities.

Integral Calculus helps answer the paradox:

[Summing must be replaced by integrating]

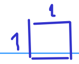
Sum $\neq$ integral

Parallel to diff. calculus: change $\neq$ rate of change.

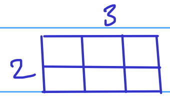
limit

Differential Calc.	Integral Calc.
• difference quotient	• sum
$\updownarrow$	$\updownarrow$
derivative	integral
• velocity, acceleration	• length, area, volume
	• almost any quantity that can be defined as limit of a sum.

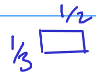
Ex:



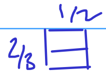
area of square of side length 1 (m) is 1 (m^2)



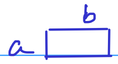
area = 6



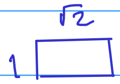
area $= \frac{1}{6}$



area $= 2 \times \frac{1}{6} = \frac{1}{3}$

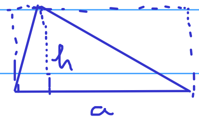


area $= a \times b$ (a, b are rational numbers)



area $= \underbrace{1 \times \sqrt{2}}$ $\quad (\sqrt{2} = \lim_{n \to \infty} a_n,\ a_n \text{ rational})$

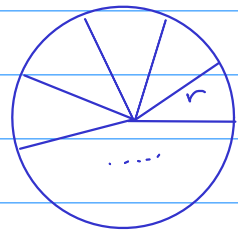
calculus involved

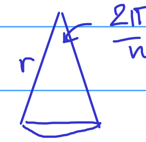


area $= \frac{1}{2} a h$

Area of circle? Archimedes (400s BC)

partition the disk into wedges.



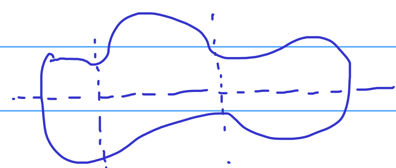


area of triangle $= r^2 \sin\frac{\pi}{n}$

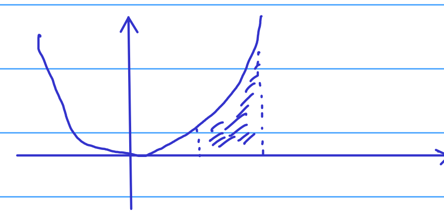
area of circle $\approx n r^2 \sin\frac{\pi}{n} \longrightarrow r^2\pi$ as $n\to\infty$

(recall: $\lim_{x\to 0}\frac{\sin x}{x} = 1$)

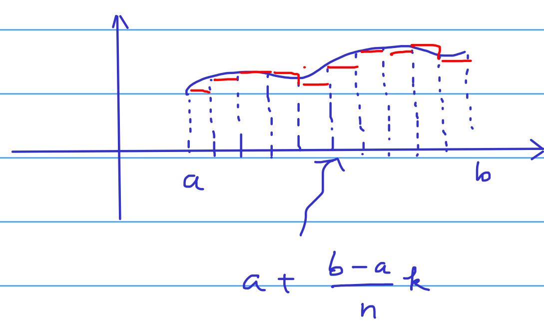
How about other shapes?



simplify



* Idea of Riemann (1850s):



area under curve

$\approx \sum$ area of rectangles

$$= \sum \underbrace{f\left(a+\frac{b-a}{n}k\right)}_{\text{height}}\underbrace{\frac{b-a}{n}}_{\text{width}}$$

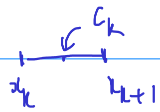
The sum: $\displaystyle\sum_{k=0}^{n-1} f\left(a+\frac{b-a}{n}k\right)\frac{b-a}{n}$ is called a Riemann sum.

or more specifically left Riemann sum.

The sum $\displaystyle\sum_{k=1}^{n} f\left(a+\frac{b-a}{n}k\right)\frac{b-a}{n}$ is called right Riemann sum.

or $\displaystyle\sum_{k=0}^{n-1} f\left(a+\frac{b-a}{n}(k+1)\right)\frac{b-a}{n}$

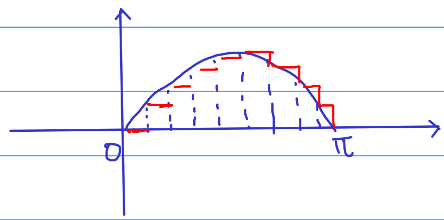
Denote $x_k = a + \frac{b-a}{n} k$



The sum $\displaystyle\sum_{k=0}^{n-1} f\left(\frac{x_k+x_{k+1}}{2}\right)\frac{b-a}{n}$ is called midpoint Riemann sum

Ex:

$f(x) = \sin x, \quad x \in [0, \pi]$



Use Mathematica to plot:

```
f[x_] := Sin[x]   (define f)
Plot[f[x], {x, 0, Pi}]
```

one can add option Filling → Axis

Left Riemann sum:

n = 10

Sum[f[$\frac{\pi}{n}$*k]*$\frac{\pi}{n}$, {k, 0, n-1}]