Project A

Due March 6, 2019

# I Instruction.

The purpose of this project is to apply integrals to

- Compute the height of water in a tank,
- Compute the volume of a spherical slice,
- Determine how fast a sum grows to infinity.

Section II is only for practice. Please do not include it in your report. Section III (the three problems) is what you write a report on. The recommended order to work on this project is:

$$\text{Problem 1} \to \text{Problem 2} \to \text{Practice} \to \text{Problem 3}.$$

If you use Mathematica for any part of the report, please write the (main) commands that you use. You don't need to write all commands. **The report should be written coherently.** You should explain the methods/procedures that you use. You can use only results/methods you already learned from the course (or previous courses). Writing only Mathematica commands without any explanation is not sufficient. Answers that are not supported by valid arguments will receive little or no credits. **You are strongly encouraged to work with your group members.**

# II Practice.

For each $n \geq 1$, denote $S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2$. In other words, $S_n$ is the sum of the square of the first $n$ positive integers. For example, $S_1 = 1$, $S_2 = 5$, $S_3 = 14$, $S_4 = 30$, etc. It is easy to see that $S_n \to \infty$ as $n \to \infty$. The question is how fast $S_n$ goes to infinity. The logarithm function $\ln n$ grows more slowly than $n$, which grows more slowly than $n^2$, which grows more slowly than exponential function $e^n$, which grows more slowly than the factorial function $n!$, etc. To know how fast a sequence goes to infinity is important in many applications. If one knows the formula:

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}, \tag{1}$$

the answer would be simple: $S_n$ goes to the infinity at the rate of $n^3$. More specifically,

$$\lim_{n \to \infty} \frac{S_n}{n^3} = \frac{1}{3}. \tag{2}$$

However, such a nice formula as (1) is rare. How can one still achieve (2) without using the identity (1)? Definite integral provides a means to do so. First, we write $S_n$ using $\Sigma$ notation:

$$S_n = \sum_{k=1}^{n} k^2.$$

The goal is to relate $S_n$ to the Riemann sum of a suitable function. Now divide $S_n$ by $n^3$ in a somewhat clever way:

$$\frac{S_n}{n^3} = \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} \frac{k^2}{n^3} = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n}$$

This is the Riemann sum of function $f(x) = x^2$ on the interval $[0, 1]$ with grid-points:

$$x_0 = \frac{0}{n},\ x_1 = \frac{1}{n},\ x_2 = \frac{2}{n}, \ldots, x_n = \frac{n}{n}$$

and sample points: $x_k^* = x_k$, which is the right endpoint of the $k$'th subinterval. The limit of this Riemann sum as $n \to \infty$ is the definite integral of $f$ on $[0, 1]$. Therefore,

$$\lim_{n\to\infty} \frac{S_n}{n^3} = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$$

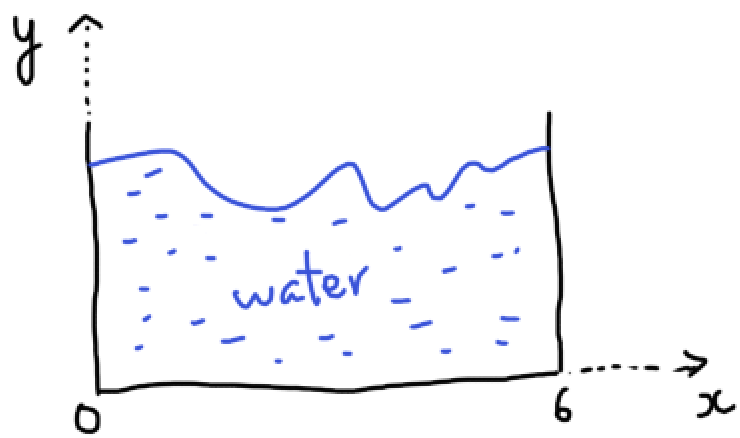
## III Problems.

1. A tank containing water is being transferred from on place to another. Looking from one side, one sees that the tank has rectangular shape. The width of the tank extends along the $x$-axis from 0 to 6 feet. The water level in the tank is measured from the bottom of it. The water is assumed to be incompressible. This means that during transfer, the water moves but the area it occupies does not change. Suppose at some point one observes that (by taking a picture for example) the water level is given by function:

$$f(x) = 3 - \frac{3}{10}\sin x - \frac{1}{10}x\cos x$$

   (a) Use Mathematica to graph the water level. Use the option Filling in the Plot command to shade the area the water occupies.

   (b) After the tank is well settled on flat ground, what is the water level?

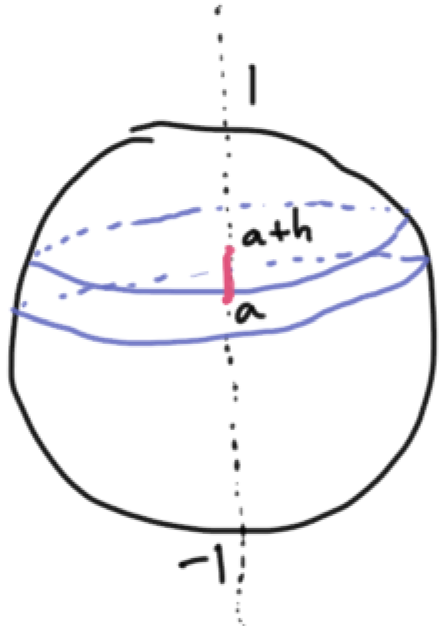



2. Consider the unit solid sphere $S$ (i.e. sphere with radius 1). The sphere extends on its vertical axis from $-1$ to 1. We cut a slice from $S$ by two planes parallel to each other and perpendicular to the vertical axis. The lower plane intersects the vertical axis at $a \in [-1, 1]$, the upper plane at $a + h$. You can think of it as a slice cut from an orange, where $h$ is the thickness of the slice, and $a$ is the position (of the lower base) of the slice. See picture.

   (a) Compute the volume of this slice in terms of $h$ and $a$.

   (b) Does this volume depend on $a$? What do you think is a practical application of this observation?

3. For each $n \geq 1$, denote $S_n = \sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n}$. In other words, $S_n$ is the sum of the square root of the first $n$ positive integers.




(a) Evaluate $S_1$, $S_2$,..., $S_6$.

(b) Use Mathematica to evaluate the quotient $\frac{S_n}{n^{3/2}}$ for $n = 5$, 10, 50, 200, 1000, 10000. What is your guess for $\lim_{n\to\infty} \frac{S_n}{n^{3/2}}$?

(c) Use a similar procedure as in the Practice to find $\lim_{n\to\infty} \frac{S_n}{n^{3/2}}$.

(d) What can you say about how fast $S_n$ goes to infinity as $n$ goes to infinity?