MATH 252H, SECTION 1, MIDTERM EXAM, WINTER 2019

Name	Student ID

- Read the instruction of each problem carefully.
- The exam has 6 pages. **Circle your final results.** Provided at the bottom of this cover page are some helpful formulae.
- To get full credit for a problem **you must show your work**. Answers not supported by valid arguments will get little or no credit.

Problem	Possible points	Earned points
1	10	
2	10	
3	20	
4	15	
5	30	
6	15	
Total	100	

Helpful trigonometric identities:

$$\begin{aligned}\sin a\cos b &= \frac{1}{2}\sin(a+b)+\frac{1}{2}\sin(a-b)\\ \sin a\sin b &= \frac{1}{2}\cos(a-b)-\frac{1}{2}\cos(a+b)\\ \cos a\cos b &= \frac{1}{2}\cos(a-b)+\frac{1}{2}\cos(a+b)\end{aligned}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

Derivative of inverse trigonometric functions:

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \quad (\arctan x)' = \frac{1}{1+x^2}$$

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Problem 1. (10 pts) Express the following sum using Σ notation:

$$1 + 3 + 5 + 7 + \ldots + 99$$

$$\sum_{k=1}^{50} (2k-1) \quad \text{or} \quad \sum_{k=0}^{49} (2k+1)$$

There are many other possible answers.

Problem 2. (10 pts) Evaluate the following sum:

$$\sum_{k=1}^{5} (2k+1)$$

$$= (2(1)+1) + (2(2)+1) + (2(3)+1) + (2(4)+1) + (2(5)+1)$$

$$= 3 + 5 + 7 + 9 + 11$$

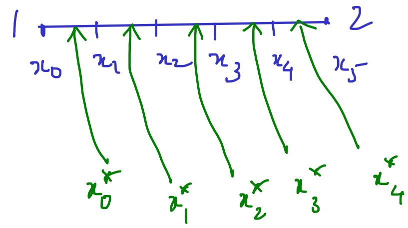
$$= 35$$

Problem 3. Consider the function $f(x) = \sqrt{x}$ on the interval [1, 2]. Suppose one wants to write the **midpoint** Riemann sum of f on this interval with $n = 5$.

(a) (3 ts) Find the width of each subinterval.

$$\frac{2-1}{5} = \frac{1}{5} = 0.2$$

(b) (6 pts) Find the grid-points (the x_k's).



$x_0 = 1$

$x_1 = 1.2$

$x_2 = 1.4$

$x_3 = 1.6$

$x_4 = 1.8$

$x_5 = 2$

(c) (6 pts) Find the sample points (the x_k^*'s).

$x_0^* = 1.1$

$x_1^* = 1.3$

$x_2^* = 1.5$

$x_3^* = 1.7$

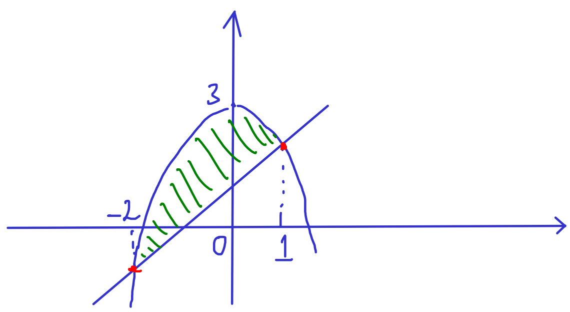
$x_4^* = 1.9$

(d) (5 pts) Write the midpoint Riemann sum of f. The sum should not contain Σ sign nor the dots "..." It should contain only numbers. Do not evaluate the sum.

$$f(x_0^*)\,0.2 + f(x_1^*)\,0.2 + f(x_2^*)\,0.2 + f(x_3^*)\,0.2 + f(x_3^*)\,0.2$$

$$= \sqrt{1.1}\;0.2 + \sqrt{1.3}\;0.2 + \sqrt{1.5}\;0.2 + \sqrt{1.7}\;0.2 + \sqrt{1.9}\;0.2$$

Problem 4. Consider the curves $y = 3 - x^2$ and $y = x + 1$.
(a) (3 pts) Graph these curves on the same graph.



(b) (6 pts) Find the points of intersection of these curves.

$3 - x^2 = x + 1 \iff x^2 + x - 2 = 0$

two roots: $x = 1$ and $x = -2$

For $x = 1$: $y = 3 - (1)^2 = 2$

For $x = -2$: $y = 3 - (-2)^2 = -1$

Intercepts: $(1, 2)$

$(-2, -1)$

(c) (6 pts) Shade the region enclosed by the curves. Then evaluate the area of this region.

$$\int_{-2}^{1} \left((3-x^2)-(x+1)\right) dx = \int_{-2}^{1} (2-x^2-x)\, dx$$

$$= \left(2x - \frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_{-2}^{1}$$

$$= \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - 2\right)$$

$$= \frac{5}{3} - \frac{1}{2} + 6 - \frac{8}{3} = -1 - \frac{1}{2} + 6 = 5 - \frac{1}{2} = \frac{9}{2}$$

Problem 5. Evaluate the following definite integrals:

(a) (10 pts)

$$\int_1^2 \frac{x^2+1}{x^3+3x} dx$$

$u = x^3 + 3x$

$du = (3x^2+3)\,dx$

$= 3(x^2+1)\,dx$

x	1	2
u	4	14

$$\frac{x^2+1}{x^3+3x} dx = \frac{\frac{1}{3} du}{u}$$

$$\int_1^2 \cdots = \int_4^{14} \frac{1}{3}\frac{1}{u}\, du = \frac{1}{3}\ln u \Big|_4^{14} = \frac{1}{3}(\ln 14 - \ln 4)$$

(b) (10 pts)

$$\int_0^2 \frac{3}{4+x^2}dx$$

$u = \frac{x}{2}$

$du = \frac{1}{2}dx$

x	0	2
u	0	1

$$\frac{3}{4+x^2}dx = \frac{3}{4+(2u)^2}\,2du = \frac{3}{4(1+u^2)}\,2du = \frac{3}{2}\frac{1}{1+u^2}du$$

$$\int_0^2 \cdots = \int_0^1 \frac{3}{2}\frac{1}{1+u^2}du = \frac{3}{2}\arctan u\Big|_0^1$$

$$= \frac{3}{2}(\underbrace{\arctan 1}_{\pi/4} - \underbrace{\arctan 0}_{0}) = \frac{3\pi}{8}$$

(c) (10 pts)

$$\int_0^{\pi/2} \sin^2 x \cos^3 x\, dx$$

$u = \sin x$

$du = \cos x\, dx$

x	0	$\pi/2$
u	0	1

$$\sin^2 x \cos^3 x\, dx = \underbrace{\sin^2 x}_{u^2}\,\underbrace{\cos^2 x}_{1-u^2}\,\underbrace{\cos x\, dx}_{du} = u^2(1-u^2)\,du$$

$$\int_0^{\pi/2} \cdots = \int_0^1 u^2(1-u^2)\,du = \int_0^1 (u^2 - u^4)\,du = \left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

Problem 6. (15 pts) Evaluate the length of the following curve

$$y = \ln x - \frac{x^2}{8} \text{ on } [1,\, 2]$$

$$f'(x) = \frac{1}{x} - \frac{x}{4}$$

$$f'(x)^2 = \left(\frac{1}{x} - \frac{x}{4}\right)^2 = \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{16}$$

$$1 + f'(x)^2 = \frac{1}{x^2} + \frac{1}{2} + \frac{x^2}{16} = \left(\frac{1}{x} + \frac{x}{4}\right)^2$$

$$\sqrt{1 + f'(x)^2} = \frac{1}{x} + \frac{x}{4}$$

$$L = \int_1^2 \left(\frac{1}{x} + \frac{x}{4}\right) dx = \left(\ln x + \frac{x^2}{8}\right)\Bigg|_1^2 = \ln 2 + \frac{2^2}{8} - \left(\ln 1 + \frac{1^2}{8}\right)$$

$$= \ln 2 + \frac{3}{8}$$