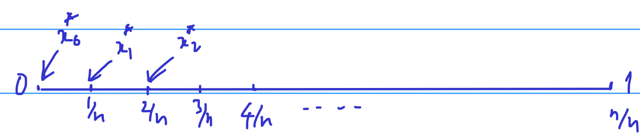
Solution to some problems in HW3

3) a. $$\sum_{k=0}^{n-1}\left(1+\frac{k^3}{n^3}\right)\frac{1}{n}$$

This is a Riemann sum of function $f(x) = 1+x^3$, with n subintervals (the number of terms in the sum), each having width $1/n$.

Height: $$1+\frac{k^3}{n^3} = f\left(\frac{k}{n}\right)$$

Sample points: $x_k^* = \frac{k}{n}$, $k = 0, 1, 2, \ldots, n-1$



Interval $[0,1]$. The sum is a left-point Riemann sum.

Therefore, by the def. of definite integrals,
$$\lim_{n\to\infty}\sum_{k=0}^{n-1}\left(1+\frac{k^3}{n^3}\right)\frac{1}{n} = \int_0^1 f(x)\,dx = \int_0^1 (1+x^3)\,dx$$

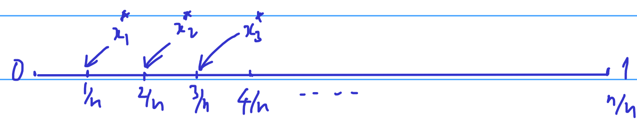
By the Fundamental Theorem of Calculus,
$$\int_0^1 (1+x^3)\,dx = \left(x+\frac{x^4}{4}\right)\Bigg|_0^1 = \left(1+\frac{1^4}{4}\right)-\left(0+\frac{0^4}{4}\right) = \frac{17}{4}$$

b. $$\sum_{k=1}^{n}\frac{1}{n}\sin\left(\frac{2k\pi}{n}\right)$$

This is a Riemann sum of function $f(x) = 1+\sin(2\pi x)$ with n subintervals (the number of terms in the sum), each having width $1/n$.

Height: $$\sin\left(\frac{2k\pi}{n}\right) = f\left(\frac{k}{n}\right)$$

Sample points: $x_k^* = \frac{k}{n}$, $k = 1, 2, \ldots, n$



Interval $[0, 1]$. The sum is a right-point Riemann sum.
Therefore,

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{n} \sin\left(\frac{2k\pi}{n}\right) = \int_0^1 f(x)\,dx = \int_0^1 \sin(2\pi x)\,dx$$

$$= \left.\frac{-\cos(2\pi x)}{2\pi}\right|_0^1$$

$$= \frac{-\cos(2\pi)}{2\pi} - \frac{-\cos 0}{2\pi}$$

$$= 0$$