Solution to selected problems in HW1

2)

$$
\begin{aligned}
2^2 - 1^2 &= 2\times 1 + 1 \\
3^2 - 2^2 &= 2\times 2 + 1 \\
+ \quad 4^2 - 3^2 &= 2\times 3 + 1 \\
&\cdots\cdots \\
(n+1)^2 - n^2 &= 2\times n + 1 \\
\hline
\end{aligned}
$$

$$\underbrace{(n+1)^2 - 1^2}_{n^2+2n} = 2(1+2+\cdots+n) + \underbrace{1+\cdots+1}_{n \text{ times}} = 2\sum_{k=1}^{n} k + n$$

Thus, $$\sum_{k=1}^{n} k = \frac{n^2+2n-n}{2} = \frac{n^2+n}{2}$$

3)

$$
\begin{aligned}
(k+1)^3 - k^3 &= (k+1)^2(k+1) - k^3 \\
&= (k^2+2k+1)(k+1) - k^3 \\
&= (k^3+3k^2+3k+1) - k^3 \\
&= 3k^2 + 3k + 1
\end{aligned}
$$

Now sum both sides over $k = 1, 2, \ldots, n$:

$$(n+1)^3 - 1^3 = 3\sum_{k=1}^{n} k^2 + 3\underbrace{\sum_{k=1}^{n} k}_{\frac{n^2+n}{2}} + \underbrace{\sum_{k=1}^{n} 1}_{n}$$

Then $$3\sum_{k=1}^{n} k^2 = (n+1)^3 - 1 - 3\,\frac{n^2+n}{2} - n$$

therefore,

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}(n+1)^3 - \frac{1}{3} - \frac{n^2+n}{2} - \frac{n}{3}.$$