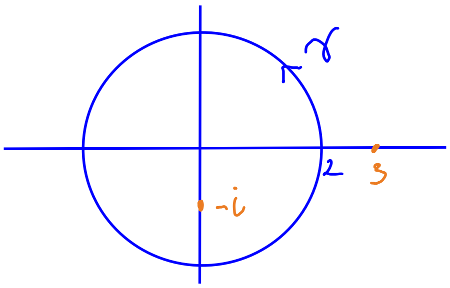
Worksheet
6/1/2020

Evaluate the following complex integrals:

(a) $\int_\gamma \frac{z+1}{(z-3)(z+i)}dz$ where γ is the the unit circle centered at the origin with radius 2 positively oriented.



The integrand $f(z) = \dfrac{z+1}{(z-3)(z+i)}$ is holomorphic everywhere except at 3 and $-i$. Only $-i$ is enclosed by γ

$$\int_\gamma f(z)\,dz = \int_\gamma \frac{\overbrace{\frac{z+1}{z-3}}^{=g(z)}}{z+i}\,dz$$

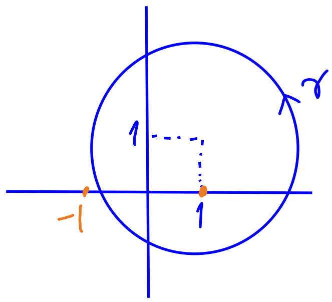
Note that g is a holomorphic function inside γ. By Cauchy's Integral formula (note that γ is simple loop positively oriented).

$$\int_\gamma \frac{g(z)}{z+i}dz = \int_\gamma \frac{g(z)}{z-(-i)}dz = 2\pi i\, g(-i)$$

$$= 2\pi i \frac{-i+1}{-i-3}$$

$$= \cdots$$

(b) $\int_\gamma \frac{z}{z^2-1}dz$
where $\gamma(t) = 1 + i + 2\cos t + 2i \sin t$ and $0 \le t \le 4\pi$.



$$\gamma(t) = 1+i + 2(\cos t + i\sin t)$$
$$= 1+i+2e^{it}$$

is the circle center at $1+i$ with radius 2.
It is positively oriented.

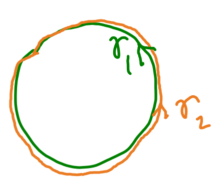
The integrand $f(z) = \dfrac{z}{z^2-1}$ is holomorphic everywhere except at $z = \pm 1$. Only 1 is enclosed inside γ.

$$\int_\gamma f(z)\,dz = \int_\gamma \frac{z}{(z-1)(z+1)}\,dz = \int_\gamma \frac{\frac{z}{z+1}}{z-1}\,dz \qquad \frac{z}{z+1} = g(z)$$

$$= \int_\gamma \frac{g(z)}{z-1}\,dz.$$

We can't apply Cauchy's Integral formula at this point because γ is <u>not</u> a simple curve. It is because γ wraps around the circle twice. It repeats itself!

We can break γ into two simple loops: $\gamma = \gamma_1 + \gamma_2$



$\gamma_1(t) = 1+i+2\cos t + 2i\sin t$
where $0 \le t \le 2\pi$

$\gamma_2(t) = 1+i+2\cos t + 2i\sin t$
where $2\pi \le t \le 4\pi$.

Note that γ_1 and γ_2 are different parametrization of the same circle.

$$\int_{\gamma} \frac{g(z)}{z-1}\,dz = \int_{\gamma_1} \frac{g(z)}{z-1}\,dz + \int_{\gamma_2} \frac{g(z)}{z-1}\,dz$$

We can now apply Cauchy's Integral Formula for each integral:

$$\int_{\gamma_1} \frac{g(z)}{z-1}\,dz = 2\pi i\, g(1) = 2\pi i \,\frac{1}{1+1} = \pi i$$

$$\int_{\gamma_2} \frac{g(z)}{z-1}\,dz = 2\pi i\, g(1) = 2\pi i \,\frac{1}{1+1} = \pi i$$

Therefore, $\displaystyle\int_{\gamma} \frac{g(z)}{z-1}\,dz = 2\pi i.$