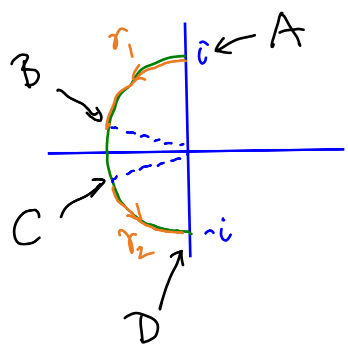
Worksheet

5/27/2020

Evaluate the following complex integrals:

(a) $\int_\gamma z^i dz$ where γ is the left half of the unit circle centered at the origin, starting at i and ending at $-i$.



$f(z) = z^i$ has antiderivative $F(z) = \dfrac{z^{i+1}}{i+1} = \dfrac{e^{(i+1)\operatorname{Log} z}}{i+1}$

on the region $G = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

Denote by γ_1 and γ_2 the curves:

$\gamma_1(t) = e^{it}$, $\quad \frac{\pi}{2} \leq t \leq \pi - \varepsilon$

$\gamma_2(t) = e^{it}$, $\quad \pi + \varepsilon \leq t \leq \pi$

See Lecture 25 for more comments.

Note that γ_1 and γ_2 lie entirely in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

We can use the fundamental theorem of calculus

(continue on next page)

(b) $\int_\gamma e^z dz$ where γ is the part of the curve $y = \sin x$ from $x = 0$ to $x = 3\pi$.

(See Lecture 24.)

(c) $\int_\gamma \frac{1}{z^2+1} dz$
where γ is the circle centered at the origin with radius 2 oriented counterclockwise. $\Big\}$ This problem will be discussed when we learn Cauchy's Integral formula.

* Continue Part (a):

$$\int_{\gamma_1} z^i dz = F(B) - F(A) = F\left(e^{i(\pi-\varepsilon)}\right) - F(i)$$

$$= \frac{\left(e^{i(\pi-\varepsilon)}\right)^{i+1}}{i+1} - \frac{i^{i+1}}{i+1}$$

$$= \frac{e^{(i+1)\operatorname{Log}\left(e^{i(\pi-\varepsilon)}\right)}}{i+1} - \frac{e^{(i+1)\operatorname{Log} i}}{i+1}$$

$$= \frac{e^{(i+1)i(\pi-\varepsilon)}}{i+1} - \frac{e^{(i+1)i\frac{\pi}{2}}}{i+1}$$

$$\xrightarrow{\varepsilon\to 0} \frac{e^{(i+1)i\pi}}{i+1} - \frac{e^{(i+1)i\frac{\pi}{2}}}{i+1} = \cdots$$

$$\int_{\gamma_2} z^i dz = F(D) - F(A) = F(-i) - F\left(e^{i(\pi+\varepsilon)}\right)$$

$$= \frac{(-i)^{i+1}}{i+1} - \frac{\left(e^{i(\pi+\varepsilon)}\right)^{i+1}}{i+1}$$

$$= \frac{e^{(i+1)\operatorname{Log}(-i)}}{i+1} - \frac{e^{(i+1)\operatorname{Log}\left(e^{i(\pi+\varepsilon)}\right)}}{i+1}$$

$$= \frac{e^{(i+1)i\left(-\frac{\pi}{2}\right)}}{i+1} - \frac{e^{(i+1)i(-\pi+\varepsilon)}}{i+1}$$

$$\xrightarrow{\varepsilon \to 0} \quad \cdots \quad - \frac{e^{(i+1)i(-\pi)}}{i+1}$$

In conclusion,

$$\int_\gamma z^i \, dz = \lim_{\varepsilon \to 0} \left(\int_{\gamma_1} + \int_{\gamma_2} \right) = \cdots\cdots$$