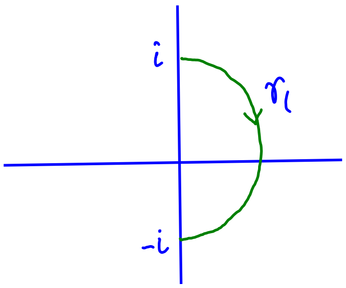
Worksheet

5/22/2020

Evaluate the following complex integrals:

(a) $\int_{\gamma_1} \frac{1}{z} dz$ where γ_1 is the right half of the unit circle centered at the origin, starting at i and ending at $-i$.



There are two methods to solve this problem.

Method 1: use parametrization of γ_1

$\gamma_1(t) = e^{-it}$ where $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.

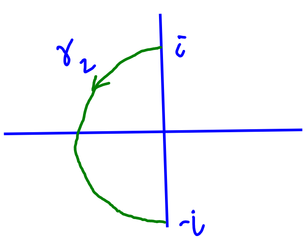
$\gamma_1'(t) = (-i) e^{-it}$

$$\int_{\gamma_1} \frac{1}{z} dz = \int_{-\pi/2}^{\pi/2} \frac{1}{e^{-it}} (-i) e^{-it} dt = \int_{-\pi/2}^{\pi/2} (-i) dt = -i\pi.$$

Method 2: use Fundamental theorem of Calculus.

(see page 3)

(b) $\int_{\gamma_2} \frac{1}{z} dz$ where γ_2 is the left half of the unit circle centered at the origin, starting at i and ending at $-i$.



Method 1: use parametrization of γ_2

$\gamma_2(t) = e^{it}$ where $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

$\gamma_2'(t) = ie^{it}$

$$\int_{\gamma_2} \frac{1}{z} dz = \int_{\pi/2}^{3\pi/2} \frac{1}{e^{it}} ie^{it} dt = \int_{\pi/2}^{3\pi/2} i dt = i\pi.$$

Method 2: use Fundamental theorem of Calculus.

(see page 3)

(c) $\int_\gamma \frac{z}{z^2+1}\,dz$
where γ is the straight line segment from to origin to $1+i$.

Put $u = z^2+1$.

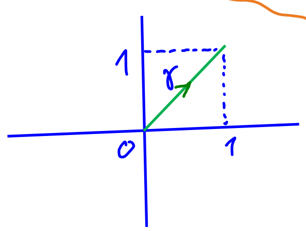
Then $du = 2z\,dz$

$$\int_\gamma \frac{z}{z^2+1}\,dz = \int_\eta \frac{1}{2}\,\frac{1}{u}\,du$$

where η is the image of the curve γ under the map z^2+1.

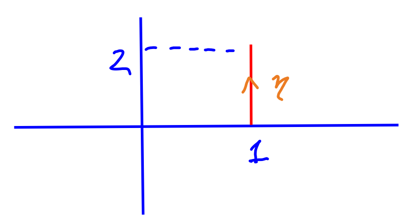
In other words,

$$\eta(t) = \gamma(t)^2+1 = (t+it)^2+1 = t^2+2it^2-t^2+1$$
$$= 1+2t^2 i$$



$\gamma(t) = x(t)+iy(t)$
$= t+it$
where $0 \le t \le 1$

We see that η is a line segment on the line $x=1$.



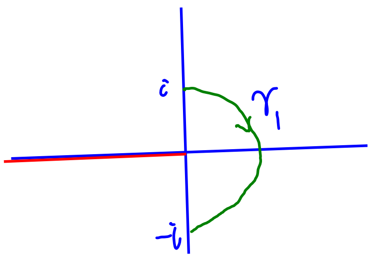
The function $f(u) = \frac{1}{2}\,\frac{1}{u}$ has an antiderivative $F(u) = \frac{1}{2}\,\mathrm{Log}(u)$

in the region $G = \mathbb{C}\setminus\mathbb{R}_{\le 0}$.

Because η lies entirely in G, we can use Fund. Thm. of Calc.

$$\int_\gamma \frac{z}{z^2+1}\,dz = \int_\eta \frac{1}{2}\,\frac{1}{u}\,du = F(\eta(1)) - F(\eta(0)) = \frac{1}{2}\,\mathrm{Log}(1+2i) - \frac{1}{2}\,\mathrm{Log}(1)$$
$$= \cdots$$

(a) Method 2: use Fundamental Thm of Calculus.



$f(z) = \frac{1}{z}$ has antiderivative $F(z) = \mathrm{Log}\, z$ on the region $G = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

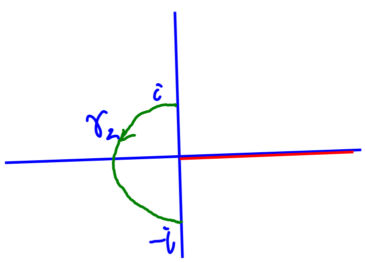
We see that γ_1 lies entirely in G.

By Fund. Thm. of Calc., we have

$$\int_{\gamma_1} f(z)\,dz = F(-i) - F(i) = \mathrm{Log}(-i) - \mathrm{Log}(i)$$

$$= -i\pi.$$

(b) Method 2: use Fundamental Thm of Calculus.



$f(z) = \frac{1}{z}$ has antiderivative $\tilde{F}(z) = \mathrm{Log}(-z)$ on the region $G = \mathbb{C} \setminus \mathbb{R}_{\geq 0}$.

We see that γ_2 lies entirely in G.

By Fund. Thm. of Calc., we have

$$\int_{\gamma_2} f(z)\,dz = \tilde{F}(-i) - \tilde{F}(i) = \mathrm{Log}(i) - \mathrm{Log}(-i)$$

$$= i\pi.$$