Worksheet

5/11/2020

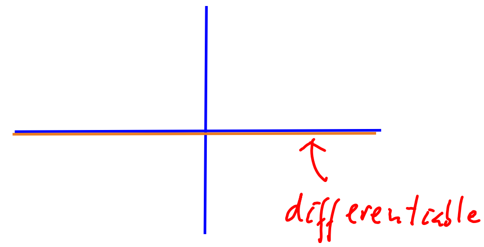
1. Find the region where function f is differentiable. Where is it holomorphic? Find the derivative of f.

   (a) $f(z) = \underbrace{x^2 + y^2}_{u(x,y)} + i\underbrace{2xy}_{v(x,y)}$

We have

$$u_x = 2x \qquad u_y = 2y$$

$$v_y = 2x \qquad v_x = 2y$$



The points $z = x+iy$ where f is differentiable are those where

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

This system becomes $y = 0$ and x is arbitrary. Thus, f is differentiable on the real line and not diff. everywhere else. It is nowhere holomorphic.

$$f'(z) = u_x + iv_x = 2x + i2y = 2(x+iy) = 2x.$$

   (b) $f(z) = \underbrace{x^3 - 3xy^2}_{u(x,y)} + i(\underbrace{3x^2y - y^2}_{v(x,y)})$

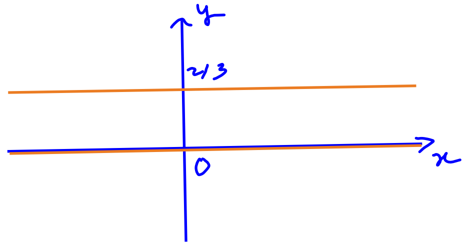
$$u_x = 3x^2 - 3y^2, \qquad v_y = 3x^2 - 2y$$

$$u_y = -6xy, \qquad v_x = 6xy$$

The system

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{becomes} \quad -3y^2 = -2y$$

This gives $y = 0$ and $y = \frac{2}{3}$.

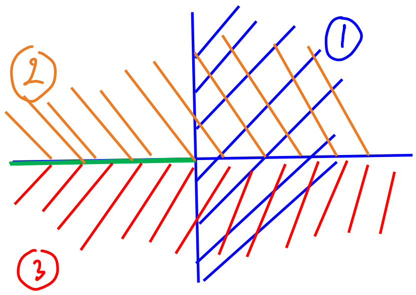


In conclusion, f is differentiable on the lines $y=0$ and $y = 2/3$. It is nowhere holomorphic.

$$f'(z) = u_x + iv_x = (3x^2 - 3y^2) + i(6xy)$$

(c) $f(z) = \operatorname{Log} z$.
Hint: Use Cauchy-Riemann equation and the fact that

$$\operatorname{Arg} z = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } \ x > 0, \\ \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right) & \text{if } \ y > 0, \\ -\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right) & \text{if } \ y < 0, \end{cases}$$



We know that f is discontinuous on the negative real line $\mathbb{R}_{\leq 0}$, so it is not differentiable there

$$f(z) = \ln|z| + i \operatorname{Arg} z$$
$$= \ln\sqrt{x^2+y^2} + i \operatorname{Arg} z$$
$$= \underbrace{\frac{1}{2}\ln(x^2+y^2)}_{u(x,y)} + i\underbrace{\operatorname{Arg} z}_{v(x,y)}.$$

We have
$$u_x = \frac{1}{2}\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2},$$

$$u_y = \frac{1}{2}\frac{2y}{x^2+y^2} = \frac{y}{x^2+y^2}$$

In domain (1): $x > 0$.

$$v(x,y) = \arctan\left(\frac{y}{x}\right)$$

$$v_x = \frac{-y/x^2}{1+\left(\frac{y}{x}\right)^2} = \frac{-y}{x^2+y^2}$$

$$v_y = \frac{1/x}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2}$$

Cauchy-Riemann eq. are satisfied for any $(x,y) \in$ (1).

Similarly, one can check that the Cauchy-Riemann eqs. are satisfied in (2) and (3).

Therefore, f is differentiable anywhere in $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$. It is holomorphic on $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$.

$$f'(z) = u_x + iv_x = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{x-iy}{x^2+y^2}.$$

One can further simplify this expression as follows:

$$f'(z) = \frac{x-iy}{(x-iy)(x+iy)} = \frac{1}{x+iy} = \frac{1}{z}.$$