Worksheet

5/1/2020

1. Write each of the following complex numbers in standard form $a + ib$ where a and b are numerical values. Round to 4 digits after the decimal point.

   (a) $\text{Log}(e^{i\frac{19\pi}{4}})$

   (b) $\left|\frac{(i+1)^{10}}{(2i+1)^8}\right|$

   (c) $(2i-1)^{i+1}$ (principal logarithm is used)

   (d) $\text{Arg}(e^{i+\sqrt{2}})$

2. Find all $z \in \mathbb{C}$ that satisfy the equation

$$\tan z = 1$$

3. Show the following trigonometric identities:

   (a) $\sin^2 z + \cos^2 z = 1$

   (b) $\sin^2 z = \frac{1-\cos(2z)}{2}$

   (c) $\cos^2 z = \frac{1+\cos(2z)}{2}$

   (d) $\sin z + \cos z = \sqrt{2}\sin(z + \frac{\pi}{4})$

4. To each of the following functions, determine what curve(s) on the complex plane should be removed so that the function is continuous.

   (a) $\frac{z+1}{z^3+i}$

   (b) $\sqrt{z^2-1}$ (principal logarithm is used)

   (c) $\sqrt{z+1}\sqrt{z-1}$ (principal logarithm is used)

5. Find the following limits:

   (a)

$$\lim_{z \to \frac{\pi}{3}} \frac{\cos z}{\sin z + 1}$$

   (b)

$$\lim_{z \to \infty} \frac{1}{\sin z}$$

   (c)

$$\lim_{z \to 2i} Log(z^2 + 1)$$

6. Solve for all complex roots of the cubic equation

$$z^3 + 3z^2 - 6z - 1 = 0$$

   How many real roots does it have?

① (a) $\mathrm{Log}\left(e^{i\frac{19}{4}\pi}\right) = \underbrace{\ln 1}_{=0} + i\,\underbrace{\mathrm{Arg}\left(e^{i\frac{19\pi}{4}}\right)}$

$= \frac{19\pi}{4} + k2\pi$ where k is chosen such that $\frac{19\pi}{4} + k2\pi \in (-\pi, \pi]$.

The only integer k that satisfies this condition is $k = -2$.

$$\frac{19\pi}{4} + (-2)\,2\pi = \frac{3\pi}{4}.$$

Therefore,

$$\mathrm{Log}\left(e^{i\frac{19\pi}{4}}\right) = i\,\frac{3\pi}{4} \approx 0 + 2.3562\,i$$

(b) $$\left|\frac{(i+1)^{10}}{(2i+1)^{8}}\right| = \frac{|(i+1)^{10}|}{|(2i+1)^{8}|} = \frac{|i+1|^{10}}{|2i+1|^{8}} = \frac{\left(\sqrt{1^2+1^2}\right)^{10}}{\left(\sqrt{2^2+1^2}\right)^{10}}$$

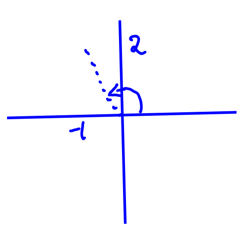
$$= \frac{(\sqrt{2})^{10}}{(\sqrt{5})^{8}} = \frac{2^5}{5^4} = \frac{32}{625} = 0.0512 + 0i$$

(c) $$(2i-1)^{i+1} = e^{(i+1)\mathrm{Log}(2i-1)}$$

$$= e^{(i+1)(a+bi)}$$

where $a = \ln|2i-1| = \ln\sqrt{5} \approx 0.8047$

$b = \mathrm{Arg}(2i-1) = \arctan\left(\frac{-1}{2}\right) + \pi \approx 2.6779.$



Continue: $(2i-1)^{i+1} = e^{a-b+i(a+b)}$

$$= e^{a-b}\cos(a+b) + i\,e^{a-b}\sin(a+b)$$

$$\approx \;\ldots + i\,\ldots$$

(d) $\operatorname{Arg}(e^{i+\sqrt{2}}) = \operatorname{Arg}(\underbrace{e^{\sqrt{2}}e^{i}}) = 1.$

polar form $re^{i\theta}$
with $r = e^{\sqrt{2}}$ and $\theta = 1$

4 (a) $f(z) = \dfrac{z+1}{z^3+i}$

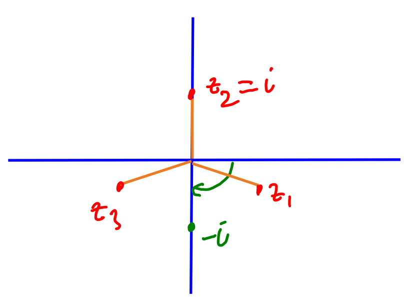
This function is a rational function (i.e. quotient of two polynomials). It is continuous everywhere except at points z where

$$z^3 + i = 0.$$

We solve for these values of z:

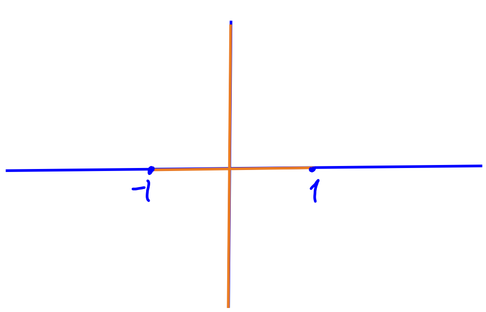
$$z^3 = -i$$

$$\rightsquigarrow \quad z = \sqrt[3]{-i} = \sqrt[3]{1 \cdot e^{-\frac{\pi}{2}i}} = e^{\left(-\frac{\pi}{6} + k\frac{2\pi}{3}\right)i} \quad \text{for } k = 0, 1, 2.$$



The function f is continuous everywhere except at z_1, z_2, z_3.

(b) See solution on worksheet 4/29.



One should remove the orange lines.

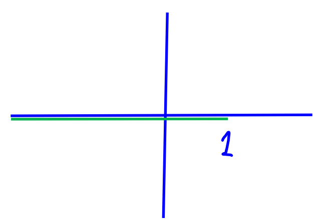
(c) $f(z) = \sqrt{z-1}\,\sqrt{z+1}.$

f is the product of

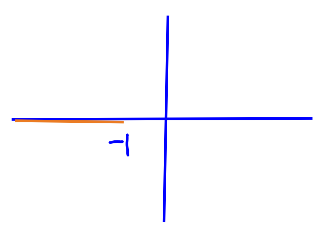
$$g(z) = \sqrt{z-1} = e^{\frac{1}{2}\operatorname{Log}(z-1)},$$

$$h(z) = \sqrt{z+1} = e^{\frac{1}{2}\operatorname{Log}(z+1)}.$$

g is continuous everywhere except at those points z such that $z-1 \in \mathbb{R}_{\leq 0}$. To know where those points are, we write $z-1 = t$ where $t \in \mathbb{R}$, $t \leq 0$. Then $z = t+1$, which is any real number ≤ 1.



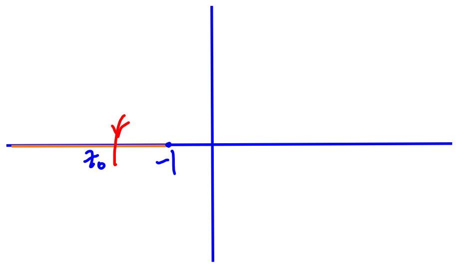
Similarly, h is continuous everywhere except at those points z such that $z+1 \in \mathbb{R}_{\leq 0}$. They are real numbers ≤ -1.



Take the union of the "bad" z's (i.e. the points at which either g or h is discontinuous): $\mathbb{R}_{\leq 1}$.

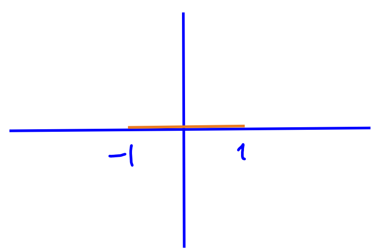
Conclusion: f is continuous anywhere in $\mathbb{C} \setminus \mathbb{R}_{\leq -1}$.

Note: It is possible that f is continuous at certain points on $\mathbb{R}_{\leq -1}$ because the jump of g may be compensated by the jump of h. In fact, one can check (geometrically) that: at any point $z_0 \in \mathbb{R}_{<-1}$,



when one crosses the real axis, g is suddenly scaled by $e^{i\pi} = -1$ and h is also suddenly scaled by $e^{i\pi} = -1$.

The product $g(z)\,h(z)$ doesn't suddenly change its sign. One can conclude from here that $f(z)$ is continuous anywhere on $\mathbb{C} \setminus [-1, 1]$ (see picture below)



The problem only asks us to remove some curves to make $f(z)$ continuous. It doesn't ask us to determine the region of continuity (i.e. the set of <u>all</u> points z where f is continuous), so the analysis written in green ink is not necessary.