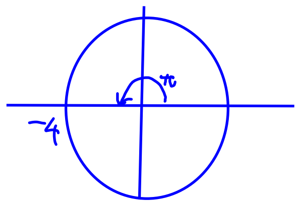
Worksheet

4/8/2020

1. Solve for complex roots of the equation $z^2 + 2z + 2 = 0$.

By the quadratic formula, $z = \dfrac{-2 + \sqrt{2^2 - 4\times 2}}{2} = \dfrac{-2+\sqrt{-4}}{2}$.

We have $-4 = 4e^{i\pi}$. Thus,



$$\sqrt{-4} = 2e^{i\left(\frac{\pi}{2} + k\frac{2\pi}{2}\right)} = 2e^{i\left(\frac{\pi}{2}+k\pi\right)} \quad \text{for } k = 0, 1$$
$$= \pm 2i$$

Therefore,

$$z = \frac{-2 \pm 2i}{2} = -1 \pm i$$

2. Solve for complex roots of the equation $z^3 - 2z^2 + 2z - 2 - i = 0$, knowing that $z = i$ is one of the roots.

Because $z = i$ is a root, we know that $z - i$ is a factor of $z^3 - 2z^2 + 2z - 2 - i$. We do long division!

$$z^2 + (-2+i)z + 1 - 2i$$
$$z - i \;\big)\; z^3 - 2z^2 + 2z - 2 - i.$$
$$-(z^3 - iz^2)$$
$$(-2+i)z^2 + 2z - 2 - i$$
$$-\left[(-2+i)z^2 - i(-2+i)z\right]$$
$$(1-2i)z - 2 - i$$
$$-\left[(1-2i)z - i(1-2i)\right]$$
$$0$$

Thus, $z^3 - 2z^2 + 2z - 2 - i = (z-i)\left(z^2 + (-2+i)z + 1 - 2i\right)$

Now we only need to solve for the roots of the quadratic equation

$$z^2 + (-2+i)z + 1 - 2i = 0.$$

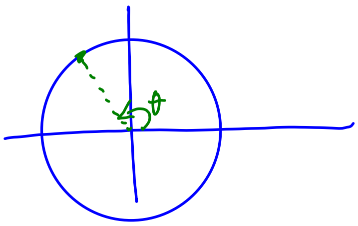
$$\Delta = (-2+i)^2 - 4(1-2i) = 4 + \underbrace{i^2}_{-1} - 4i - 4 + 8i = -1 + 4i.$$

In polar form,

$$\Delta = r e^{i\theta} \quad \text{with} \quad r = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$$

and

$$\begin{cases} \cos\theta = -1/\sqrt{17} \\ \sin\theta = 4/\sqrt{17} \end{cases}$$



$$\theta = \arccos\left(-\frac{1}{\sqrt{17}}\right) = 1.81577....$$

Then $\sqrt{\Delta} = \sqrt{r}\, e^{i\left(\frac{\theta}{2} + k\pi\right)}$ (with $k = 0, 1$)

$$= \pm \sqrt[4]{17}\, e^{i\, 1.81577...}$$

$$\approx \pm \sqrt[4]{17} \left(\cos 1.81577 + i \sin 1.81577\right)$$

$$\approx \pm (-0.4925 + 1.9699\, i)$$

The roots of the quadratic equation are

$$z \approx \frac{-(-2+i) \pm (-0.4925 + 1.9699\, i)}{2}$$

Thus,

$$z_1 \approx 0.75375 + i\, 0.48495$$

$$z_2 \approx 1.2463 - i\, 1.4849$$

In conclusion, the eq. $z^3 - 2z^2 + 2z - 2 - i = 0$ has three roots:

z_1, z_2 (as above) and $z_3 = i$.

3. Solve for complex roots of the equation $z^4 + z^2 + 1 = 0$.

Put $w = z^2$. We get $w^2 + w + 1 = 0$. $(*)$

Then
$$w = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

The equation $(*)$ has two solutions

$$w_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\frac{2\pi}{3}}$$

$$w_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{i\left(-\frac{2\pi}{3}\right)}.$$

Now we solve for z such that $z^2 = w_1$. That is to take the square roots of w_1.

$$z = \sqrt{w_1} = \pm e^{i\frac{\pi}{3}}.$$

Then we take the square roots of w_2.

$$z = \sqrt{w_2} = \pm e^{-i\frac{\pi}{3}}.$$

Therefore, the equation $z^4 + z^2 + 1 = 0$ has four roots

$$z_1 = e^{i\pi/3}$$

$$z_2 = -e^{i\pi/3}$$

$$z_3 = e^{-i\pi/3}$$

$$z_4 = -e^{-i\pi/3}$$