Worksheet

4/29/2020

To each of the following functions, determine what curve(s) on the complex plane should be removed so that the function is continuous.

(a) 2^z (principal logarithm is used)

$f(z) = 2^z = e^{z\text{Log}2}$, which is a composite map

$$\begin{tikzcd} z \arrow[r, "\text{multiplication by a constant}"'] & z\text{Log}2 \arrow[r, "\text{exponential function}"'] & e^{z\text{Log}2} \end{tikzcd}$$

Because each component function is continuous everywhere on $\mathbb{C}$, the composite function is also continuous everywhere on $\mathbb{C}$.

Conclusion: no points need to be removed.

(b) $\sqrt{z^2-1}$ (principal logarithm is used)

$$f(z) = (z^2-1)^{1/2} = e^{\frac{1}{2}\operatorname{Log}(z^2-1)}.$$

This is a composite function:

$$\begin{tikzcd}
z \arrow[r, "\text{polynomial}", "\text{continuous}"'] & z^2-1 \arrow[r, "\operatorname{Log}"] & \operatorname{Log}(z^2-1) \arrow[r, "\text{multi. by}", "\substack{\text{constant}\\ \text{continuous}}"'] & \frac{1}{2}\operatorname{Log}(z^2-1) \arrow[d, "\exp"', "\text{continuous}"] \\
 & & & e^{\frac{1}{2}\operatorname{Log}(z^2-1)} = f(z).
\end{tikzcd}$$

The only places where f could possibly be discontinuous are points z on the complex plane such that $z^2-1 \in \mathbb{R}_{\le 0}$. With $z = x+iy$, we have

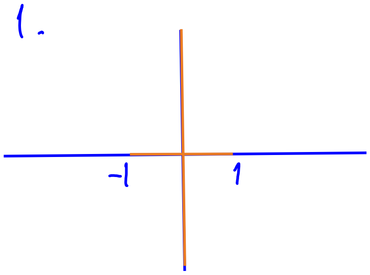
$$z^2-1 = (x+iy)^2-1 = (x^2-y^2-1)+i2xy.$$

For this point to be on $\mathbb{R}_{\le 0}$, we need

$$\begin{cases} x^2-y^2-1 \le 0 & (*) \\ 2xy = 0. & \rightsquigarrow \text{either } x=0 \text{ or } y=0 \end{cases}$$

If $x=0$ then the inequality $(*)$ becomes $-y^2-1\le 0$. This is true for any $y \in \mathbb{R}$.

If $y=0$ then the inequality $(*)$ becomes $x^2-1\le 0$. This is true for $-1\le x\le 1$.



Therefore, if we remove every point on the imaginary axis and the segment $[-1,1]$ on the real axis, we will get a continuous function. In other words, f is continuous on $\mathbb{C}\setminus(\{z : \operatorname{Re} z = 0\}\cup\{z : \operatorname{Im} z = 0, -1\le \operatorname{Re} z \le 1\})$