Worksheet

4/20/2020

1. Write the following complex numbers in either polar or standard form.

   (a) $\arcsin(\frac{1}{2})$

$$\arcsin\left(\tfrac{1}{2}\right) = \frac{1}{i}\log\left(i\frac{1}{2} + \sqrt{1-\left(\tfrac{1}{2}\right)^2}\right) = \frac{1}{i}\log\left(\frac{i}{2} \pm \frac{\sqrt{3}}{2}\right)$$

complex square root

real square root

• Plus sign: $\frac{1}{i}\log\left(\frac{i}{2} + \frac{\sqrt{3}}{2}\right) = \frac{1}{i}\left(\ln 1 + i\left(\frac{\pi}{6} + k2\pi\right)\right)$

$$= \frac{\pi}{6} + k2\pi \qquad (k \in \mathbb{Z})$$

• Minus sign: $\frac{1}{i}\log\left(\frac{i}{2} - \frac{\sqrt{3}}{2}\right) = \frac{1}{i}\left(\ln 1 + i\left(\frac{5\pi}{6} + \ell 2\pi\right)\right) = \frac{5\pi}{6} + \ell 2\pi \; (\ell \in \mathbb{Z})$.

Therefore, $\arcsin\left(\frac{1}{2}\right) = \left\{\frac{\pi}{6} + k2\pi : k \in \mathbb{Z}\right\} \cup \left\{\frac{5\pi}{6} + \ell 2\pi : \ell \in \mathbb{Z}\right\}$

   (b) Arcsin(2)

$$\operatorname{Arcsin} 2 = \frac{1}{i}\operatorname{Log}\left(i2 + \sqrt{1-2^2}\right) = \frac{1}{i}\operatorname{Log}\left(i2 + \sqrt{-3}\right).$$

using principal logarithm

$$\sqrt{-3} = (-3)^{1/2} = e^{\frac{1}{2}\operatorname{Log}(-3)} = e^{\frac{1}{2}(\ln 3 + i\pi)}$$

$$= e^{\frac{1}{2}\ln 3 + i\frac{\pi}{2}} = e^{\frac{1}{2}\ln 3}\underbrace{e^{i\frac{\pi}{2}}}_{i} = i\sqrt{3}.$$

Thus,

$$\operatorname{Arcsin} 2 = \frac{1}{i}\operatorname{Log}\left(i2 + i\sqrt{3}\right) = \frac{1}{i}\operatorname{Log}\left((2+\sqrt{3})i\right) = \frac{1}{i}\operatorname{Log}\left((2+\sqrt{3})e^{i\frac{\pi}{2}}\right)$$

$$= \frac{1}{i}\left(\ln(2+\sqrt{3}) + i\frac{\pi}{2}\right)$$

$$\operatorname{Arcsin} 2 = \frac{\pi}{2} - i\ln(2+\sqrt{3}).$$

2. Find a formula for $\arctan z$

We solve for w in terms of z from the equation $\tan w = z$.

$$\tan w = \frac{\sin w}{\cos w} = \frac{1}{i} \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}.$$

Put $u = e^{iw}$. Then:

$$z = \tan w = \frac{1}{i} \frac{u - u^{-1}}{u + u^{-1}} = \frac{1}{i} \frac{u^2 - 1}{u^2 + 1}.$$

From here, we get

$$u^2 - 1 = iz(u^2 + 1)$$

which leads to $$u^2 = \frac{1+iz}{1-iz}.$$

Thus, $$u = \sqrt{\frac{1+iz}{1-iz}}.$$

Complex square root (there are two of them)

Thus, $$e^{iw} = \sqrt{\frac{1+iz}{1-iz}}.$$

We get $$iw = \log \sqrt{\frac{1+iz}{1-iz}}.$$

Thus, $$\arctan z = w = \frac{1}{i} \log \sqrt{\frac{1+iz}{1-iz}}$$