Worksheet

4/15/2020

1. Write the following complex numbers in standard form.

   (a) $\log(1+i)$

$$1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2} e^{i\frac{\pi}{4}} .$$

Thus,

$$\log(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + k2\pi\right) \qquad k = 0, \pm 1, \pm 2, \ldots$$

   (b) $\mathrm{Log}(1+i)$

We know from above that

$$\log(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + k2\pi\right).$$

The only value of k that makes $\frac{\pi}{4} + k2\pi \in (-\pi, \pi]$ is $k = 0$.

Therefore,

$$\mathrm{Log}(1+i) = \ln\sqrt{2} + i\frac{\pi}{4}.$$

(c) 2^i

$$2^i = e^{i \log 2}$$

We have $2 = 2e^{0i}$, so $\log 2 = \ln 2 + i(0 + k2\pi)$

$$= \ln 2 + ik2\pi .$$

Thus, $2^i = e^{i(\ln 2 + ik2\pi)} = e^{i\ln 2 - k2\pi} = e^{-k2\pi} e^{i \ln 2}$

$$= e^{-k2\pi} \cos(\ln 2) + i\, e^{-k2\pi} \sin(\ln 2)$$

where $k \in \mathbb{Z}$

(d) $(i+1)^{2-2i}$

$$(i+1)^{2-2i} = e^{(2-2i)\log(i+1)}$$

$$= e^{(2-2i)\left(\ln\sqrt{2} + i\left(\frac{\pi}{4} + k2\pi\right)\right)}$$

$$= e^{2\ln\sqrt{2} + 2\left(\frac{\pi}{4} + k2\pi\right) + i\left(\underbrace{-2\ln\sqrt{2} + 2\left(\frac{\pi}{4} + k2\pi\right)}\right)}$$

$$= \underbrace{e^{2\ln\sqrt{2} + 2\left(\frac{\pi}{4} + k2\pi\right)}} \cos(\qquad) + i \qquad \sin(\qquad)$$

(e) $\sqrt[3]{-1-i}$ (using principal logarithm)

$$-1-i = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\, e^{-i\frac{3\pi}{4}}.$$

$$\log(-1-i) = \ln\sqrt{2} + i\left(-\frac{3\pi}{4} + k2\pi\right)$$

The value of k that makes $-\frac{3\pi}{4} + k2\pi \in (-\pi, \pi]$ is $k = 0$.

Thus, $\operatorname{Log}(-1-i) = \ln\sqrt{2} - i\frac{3\pi}{4}$

$$\sqrt[3]{-1-i} = (-1-i)^{1/3} = e^{\frac{1}{3}\operatorname{Log}(-1-i)} = e^{\frac{1}{3}\left(\ln\sqrt{2} - i\frac{3\pi}{4}\right)}$$

$$= e^{\frac{1}{3}\ln\sqrt{2}}\, e^{-i\frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}} e^{\frac{1}{3}\ln\sqrt{2}} - i\frac{1}{\sqrt{2}} e^{\frac{1}{3}\ln\sqrt{2}}.$$