Worksheet

4/10/2020

1. Write the following complex numbers in either standard form or polar form.

   (a) e^{e^i}

$$e^{i} = e^{0+1\cdot i} = \underbrace{e^{0}}_{=1} e^{1\cdot i} = \cos 1 + i \sin 1.$$

$$e^{e^{i}} = e^{\cos 1 + i\sin 1} = \underbrace{e^{\cos 1} e^{i \sin 1}}_{\text{polar form}} = e^{\cos 1}\left(\cos(\sin 1) + i \sin(\sin 1)\right)$$

$$= \underbrace{e^{\cos 1}\cos(\sin 1) + i e^{\cos 1}\sin(\sin 1)}_{\text{standard form}}.$$

   (b) $\tan(2+2i)$

$$= \frac{\sin(2+2i)}{\cos(2+2i)} = \frac{\frac{e^{i(2+2i)} - e^{-i(2+2i)}}{2i}}{\frac{e^{i(2+2i)} + e^{-i(2+2i)}}{2}} = \frac{1}{i}\,\frac{e^{i(2+2i)} - e^{-i(2+2i)}}{e^{i(2+2i)} + e^{-i(2+2i)}}$$

$$= \frac{1}{i}\,\frac{e^{-2+2i} - e^{2-2i}}{e^{-2+2i} + e^{2-2i}}$$

$$= \frac{1}{i}\,\frac{e^{-2}(\cos 2 + i\sin 2) - e^{2}(\cos(-2) + i\sin(-2))}{e^{-2}(\cos 2 + i\sin 2) + e^{2}(\cos(-2) + i\sin(-2))}$$

$$= \frac{1}{i}\,\frac{(e^{-2} - e^{2})\cos 2 + i(e^{-2} + e^{2})\sin 2}{(e^{-2} + e^{2})\cos 2 + i(e^{-2} - e^{2})\sin 2} =$$

$$\approx \frac{1}{i}\,\frac{1.50931 + i\,3.42095}{-1.56563 - i\,3.25783} \approx -0.028393 + i\,1.02384.$$

2. Solve for all complex roots of the equation $z^3 + 3z^2 + 1 = 0$.

Change of variable $x = z + 1$, or equivalently $z = x - 1$.

$$(x-1)^3 + 3(x-1)^2 + 1 = 0$$

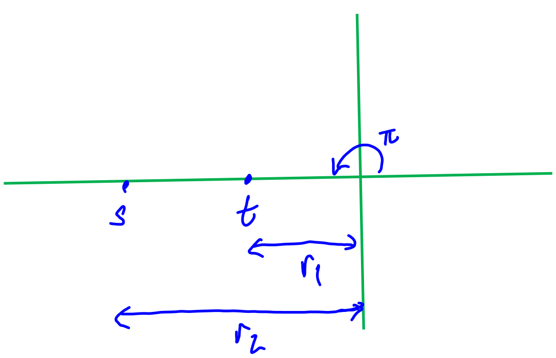
or $$x^3 - 3x + 3 = 0.$$

Here $p = -3$ and $q = 3$.

$$\Delta = q^2 + \frac{4p^3}{27} = 5.$$

$$t = u^3 = \frac{1}{2}(-q + \sqrt{\Delta}) = \frac{1}{2}(-3 + \sqrt{5}).$$

$$s = v^3 = \frac{1}{2}(-q - \sqrt{\Delta}) = \frac{1}{2}(-3 - \sqrt{5}).$$



In polar form,

$$t = -r_1 = r_1 e^{i\pi}$$

$$s = -r_2 = r_2 e^{i\pi}$$

where

$$r_1 = |t| = \frac{1}{2}(3 - \sqrt{5}) > 0,$$

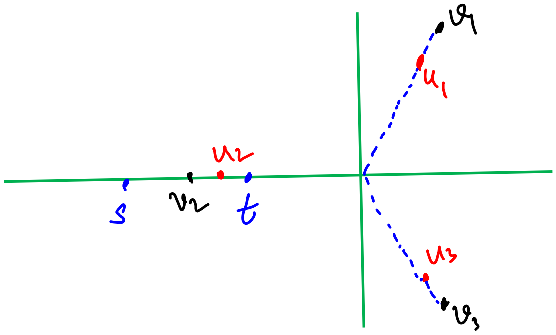
$$r_2 = |s| = \frac{1}{2}(3 + \sqrt{5}) > 0.$$

There are three values of u:

$$u_1 = \sqrt[3]{r_1}\, e^{i\frac{\pi}{3}} \approx 0.362781 + i\, 0\,628356$$

$$u_2 = \sqrt[3]{r_1}\, e^{i\left(\frac{\pi}{3} + \frac{2\pi}{3}\right)} = -\sqrt[3]{r_1} \approx -0.725563$$

$$u_3 = \sqrt[3]{r_1}\, e^{i\left(\frac{\pi}{3}+\frac{4\pi}{3}\right)} = \sqrt[3]{r_1}\, e^{i\frac{5\pi}{3}} \approx 0.362781 - i\,0.628356$$



There are also three values of v:

$$v_1 = \sqrt[3]{r_2}\, e^{i\frac{\pi}{3}}$$

$$v_2 = \sqrt[3]{r_2}\, e^{i\pi} = -\sqrt[3]{r_2} \approx 1.37824$$

$$v_3 = \sqrt[3]{r_2}\, e^{i\frac{5\pi}{3}}.$$

There are 9 combinations of u_1, u_2, u_3 and v_1, v_2, v_3. We only choose the pairs (u, v) such that $uv = -\frac{p}{3} = 1$. Because

$$\arg(uv) = \arg(u) + \arg(v) \quad (\text{in modulo } 2\pi)$$

we must have

$$\arg(u) + \arg(v) = \arg(1) = 0 \quad (\text{in modulo } 2\pi).$$

Therefore, we choose the following pairs:

$$x = u_1 + v_3 \approx \ldots\ldots$$

$$x = u_3 + v_1 \approx \ldots\ldots$$

$$x = u_2 + v_2 \approx \ldots\ldots$$

Then the roots z of the original equation are $z = x - 1 \approx \begin{cases} \ldots \\ \ldots \\ \ldots \end{cases}$