# MATH 483, MIDTERM EXAM, SPRING 2020

| Name | Student ID |
|---|---|
| | |

- Read carefully the description of each problem. You are given an extra page to work on Problem 1 and 4.

- To get full credit for each problem, you must provide valid arguments. Show in detail all steps. Use words to explain your ideas if necessary. Mysterious answers that are puzzling to the grader will receive little or no credit.

| Problem | Possible points | Earned points |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| ~~3a~~ 3 | 10 | |
| ~~3b~~ 4a | 10 | |
| ~~4~~ 4b | 10 | |
| Total | 50 | |

**Problem 1.** (10 points) Express the following complex number in standard form $a + ib$. Make sure to write the exact formula for $a$ and $b$, then round them to 4 digits after the decimal point.

$$(i+1)^{2i} + (i-1)^{2i} \quad \text{(principal logarithm is used)}$$

$$(i+1)^{2i} = e^{2i\,\mathrm{Log}(i+1)}$$

we have $i+1 = \sqrt{2}\, e^{i\frac{\pi}{4}}$. Thus, $\mathrm{Log}(i+1) = \ln\sqrt{2} + i\frac{\pi}{4}$.

Then $(i+1)^{2i} = e^{2i(\ln\sqrt{2} + i\frac{\pi}{4})} = e^{-\frac{\pi}{2} + i2\ln\sqrt{2}}$.

On the other hand,

$(i-1)^{2i} = e^{2i\,\mathrm{Log}(i-1)}$.

We have $i-1 = \sqrt{2}\, e^{i\frac{3\pi}{4}}$. Thus, $\mathrm{Log}(i-1) = \ln\sqrt{2} + i\frac{3\pi}{4}$.

Then $(i-1)^{2i} = e^{2i(\ln\sqrt{2} + i\frac{3\pi}{4})} = e^{-\frac{3\pi}{2} + i2\ln\sqrt{2}}$.

Therefore,

$$(i+1)^{2i} + (i-1)^{2i} = e^{-\frac{\pi}{2} + i2\ln\sqrt{2}} + e^{-\frac{3\pi}{2} + i2\ln\sqrt{2}}$$

$$= e^{-\frac{\pi}{2}}\cos(2\ln\sqrt{2}) + i e^{-\frac{\pi}{2}}\sin(2\ln\sqrt{2}) + e^{-\frac{3\pi}{2}}\cos(2\ln\sqrt{2}) + i e^{-\frac{3\pi}{2}}\sin(2\ln\sqrt{2})$$

$$= \underbrace{\left(e^{-\frac{\pi}{2}} + e^{-\frac{3\pi}{2}}\right)\cos 2}_{a} + i\underbrace{\left(e^{-\frac{\pi}{2}} + e^{-\frac{3\pi}{2}}\right)\sin 2}_{b}$$

By calculator,

$a \approx 0.1668$

$b \approx 0.1386$

*(Extra space if needed)*

**Problem 2.** (10 points) Prove the trigonometric identity

$$\sin(2z) = 2\sin z\cos z \quad \forall z \in \mathbb{C}.$$

By the definition of sine and cosine,

$$\sin 2z = \frac{e^{i2z} - e^{-i2z}}{2i}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Thus,

$$RHS = 2\sin z\cos z = 2\,\frac{e^{iz} - e^{-iz}}{2i}\,\frac{e^{iz} + e^{-iz}}{2}$$

$$= \frac{e^{iz}e^{iz} - \cancel{e^{-iz}e^{iz}} + \cancel{e^{iz}e^{-iz}} - e^{-iz}e^{-iz}}{2i}$$

$$= \frac{e^{2iz} - e^{-2iz}}{2i}$$

$$= LHS.$$

**Problem 3.** (10 points) Determine all complex numbers $z$ satisfying

$$\cancel{\tan z = \frac{1}{2}} \quad \frac{\tan z + 1}{\tan z - 1} = -3$$

We get $\tan z + 1 = -3(\tan z - 1)$. From here we get $\tan z = \frac{1}{2}$.

$$\tan z = \frac{1}{i}\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$$

Put $u = e^{iz}$. Then we get

$$\frac{1}{i}\frac{u - u^{-1}}{u + u^{-1}} = \frac{1}{2}$$

Equivalently,

$$\frac{u^2 - 1}{u^2 + 1} = \frac{i}{2}$$

From here we get

$$u^2 = \frac{1 + i/2}{1 - i/2} = \frac{2+i}{2-i} = \frac{(2+i)^2}{2^2+1^2} = \frac{3+4i}{5}.$$

Thus,

$$e^{i2z} = \frac{3}{5} + \frac{4}{5}i = e^{i\theta}$$

where $\theta = \arctan\left(\frac{4}{3}\right) \approx 0.9273$

Then

$$2z = \theta + k2\pi \quad \text{for } k \in \mathbb{Z}.$$

Then

$$z = \frac{\theta}{2} + k\pi \approx 0.4636 + k\pi \quad \text{where } k \in \mathbb{Z}.$$

*(Extra space if needed)*

**Problem ~~3~~ 4.** Compute the following limits.

(a)

$$\lim_{z\to 2i} \frac{(z-2i)(z+1)}{z^2+4}$$

$$\frac{(z-2i)(z+1)}{z^2+4} = \frac{\cancel{(z-2i)}(z+1)}{\cancel{(z-2i)}(z+2i)} = \frac{z+1}{z+2i}$$

$$\lim_{z\to 2i} \frac{(z-2i)(z+1)}{z^2+4} = \lim_{z\to 2i} \frac{z+1}{z+2i} = \frac{2i+1}{2i+2i} = \frac{2i+1}{4i}$$

$$= \frac{1}{2} + \frac{1}{4i} = \frac{1}{2} - i\frac{1}{4}$$

(b)

$$\lim_{z\to 0} \frac{\bar{z}^2}{z}$$

We will show that the limit is equal to zero.

Let $z_n$ be a sequence that goes to 0. We show that

$$\frac{\bar{z}_n^{\,2}}{z_n} \to 0$$

We have $$\left|\frac{\bar{z}_n^{\,2}}{z_n}\right| = \frac{|\bar{z}_n|^2}{|z_n|} = \frac{|z_n|^2}{|z_n|} = |z_n| \longrightarrow 0$$ because $z_n \to 0$.

Therefore, $$\lim_{n\to\infty} \frac{\bar{z}_n^{\,2}}{z_n} = 0.$$

We conclude that $$\lim_{z\to 0} \frac{\bar{z}^2}{z} = 0.$$