Lecture 9

Friday, April 17, 2020

Remarks on Problem 1, part (a), of HW2:

$$z^3+3z+1=0.$$

Here $p=3$ and $q=1$.

$$\Delta = q^2+\frac{4p^3}{27} = 1+4=5.$$

According to Cardano's method, we split $z=u+v$ where

$$u^3 = \frac{1}{2}\left(-q+\sqrt{\Delta}\right) = \frac{1}{2}\left(-1+\sqrt{5}\right).$$

$$v^3 = \frac{1}{2}\left(-q-\sqrt{\Delta}\right) = \frac{1}{2}\left(-1-\sqrt{5}\right).$$

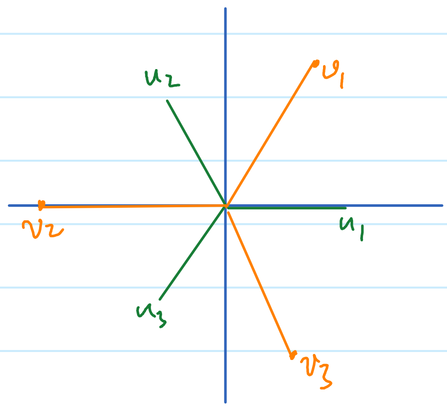
From here, we obtain three values of u and three values of v:

$$u = \sqrt[3]{\frac{1}{2}(-1+\sqrt{5})} = \sqrt[3]{r_1 e^{0i}} = \sqrt[3]{r_1}\, e^{ik\frac{2\pi}{3}}$$

where $r_1 = \frac{1}{2}(-1+\sqrt{5}) > 0$.

$$v = \sqrt{\frac{1}{2}(-1-\sqrt{5})} = \sqrt[3]{r_2 e^{\pi i}} = \sqrt[3]{r_2}\, e^{i\left(\frac{\pi}{3}+k\frac{2\pi}{3}\right)}$$

where $r_2 = \frac{1}{2}(1+\sqrt{5}) > 0$.



There are 9 combinations of u and v in total: u_1+v_1, u_1+v_2, $\dots$, u_3+v_3.

However, not all of them give us a solution to the equation

$$z^3+3z+1=0.$$

The reason is that when we convert the system

$$\begin{cases} uv = -p/3 \\ u^3+v^3 = -q \end{cases} \qquad (I)$$

into the system

$$\begin{cases} u^3v^3 = -p^3/27 \\ u^3+v^3 = -q \end{cases} \qquad (II)$$

the two systems are not equivalent. Solutions to system (I) is a solution to system (II), but not vice-versa. This is because the equation

$$x^3 = a^3$$

doesn't imply $x=a$. The implication is true when $x, a \in \mathbb{R}$, but not true when $x, a \in \mathbb{C}$. Recall that there are three third roots of a^3, only one of which is a. For example, x can be $a e^{i\frac{\pi}{3}}$, which is not equal to a.

Therefore, we only choose the combination of u and v such that $uv = -p/3$. In this problem, we need $uv = -1$. It is convenient to look at the arguments of u and v:

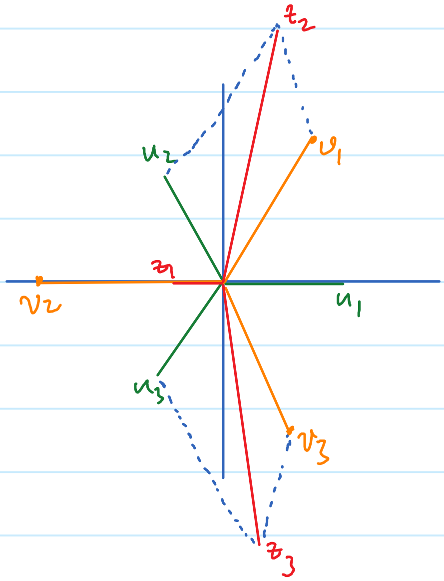
$$\arg u + \arg v = \arg(-1) \quad (\text{in modulo } 2\pi)$$
$$= \pi \quad (\text{in modulo } 2\pi).$$

From the above picture, we have

$$\arg u_1 = 0 \qquad \arg v_1 = \pi/3$$
$$\arg u_2 = \frac{2\pi}{3} \qquad \arg v_2 = \pi$$
$$\arg u_3 = \frac{4\pi}{3} \qquad \arg v_3 = \frac{5\pi}{3}$$

The correct combinations of u and v are

$$z_1 = u_1 + v_2$$
$$z_2 = u_2 + v_1$$
$$z_3 = u_3 + v_3$$



We see that the equation $z^3+3z+1=0$ has only one real root, which is z_1. The other two are complex roots. They are complex conjugate of each other.

Last time we gave the definition of log and Log.

$$\left.\begin{array}{l} \ln \\ \log \\ \mathrm{Log} \end{array}\right\} \text{ logarithm of base } e.$$

$\log z = \ln|z| + i\arg z \rightsquigarrow$ multivalued function

$\mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}\, z \rightsquigarrow$ single valued function

$\log z$ is the set of all complex numbers w such that $e^w = z$.

For example,

$$\log 3 = \ln 3 + i\arg 3 = \ln 3 + ik2\pi \quad (k\in\mathbb{Z}).$$

$\mathrm{Log}\, z$ is a special value of $\log z$.

$$\mathrm{Log}\, 3 = \ln 3 + i\,\mathrm{Arg}\, 3 = \ln 3 + i0 = \ln 3.$$

More generally,

$$\mathrm{Log}\, a = \ln a \quad \forall a\in\mathbb{R},\ a>0$$

Log is an extension of ln to complex variable. The notation $\ln(-3)$ doesn't make sense because ln is only defined for positive real numbers. There is no real number z such that $e^z=-3$. However, if z is allowed to be complex then one can find it.

$\mathrm{Log}(-3)$ is well-defined:

$$\mathrm{Log}(-3) = \ln|-3| + i\,\mathrm{Arg}(-3) = \ln 3 + i\pi.$$

$$\log(-3) = \ln|-3| + i\arg(-3) = \ln 3 + i(\pi + k2\pi) \quad (k\in\mathbb{Z}).$$

Complex powers:

We know how to take integer power of a complex number:

$$z^n = z \cdot z \cdots z \qquad \text{for } n > 0$$

$$z^{-n} = \frac{1}{z^n} = \frac{1}{z}\,\frac{1}{z} \cdots \frac{1}{z} \qquad \text{for } n > 0.$$

We also know how to take rational power of a complex number:

$$z^{1/n} = \sqrt[n]{z} = \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right)} \qquad \text{where } k = 0, 1, \dots, n-1.$$

How about complex power z^a where $a \in \mathbb{C}$?

If $z = e$ then we know how to compute $z^a = e^a$. Specifically,

$$e^a = e^{b+ic} = e^b e^{ic} = e^b(\cos c + i \sin c).$$

If $z \neq e$ then z^a is defined as

$$z^a = e^{a \log z}$$

Because $\log z$ is multivalued, z^a is in general also multivalued, except when $z = e$ or when a is an integer. Here is the process to compute z^a:

$$z \xrightarrow{\log} \log z \xrightarrow{\times a} a \log z \xrightarrow{\exp} e^{a \log z} = z^a$$

multivalued — single valued — single valued

Ex:

$$2^{\sqrt{2}} = e^{\sqrt{2} \log 2} = e^{\sqrt{2}(\ln 2 + i2k\pi)} = e^{\sqrt{2}\ln 2} e^{i2\sqrt{2}k\pi}$$

where $k \in \mathbb{Z}$.

$2^{\sqrt{2}}$ has only one real value, but infinitely many complex values.

[See more examples on the worksheet 4/15/2020]