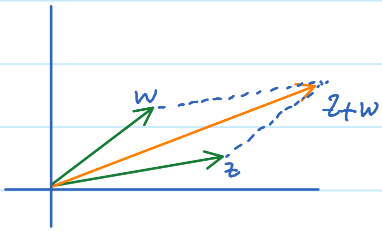
Lecture 8

Wednesday, April 15, 2020

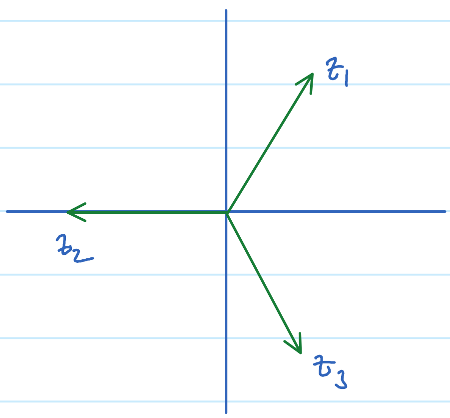
Recall that the addition of complex numbers is the same as the addition of vectors.



With $w = re^{i\theta}$, the product zw is represented by a point on the plane obtained by scaling z by factor r and then rotating by angle θ about the origin. The factor $e^{i\theta}$ represents the rotation by angle θ.

Ex :

Draw the third roots of -2 on the complex plane



In polar form, $-2 = 2(-1) = 2e^{i\pi}$.

Thus,

$$\sqrt[3]{-2} = \sqrt[3]{2}\, e^{i\left(\frac{\pi}{3} + k\frac{2\pi}{3}\right)} \quad \text{with } k \in \mathbb{Z}$$

The three third roots of 2 are

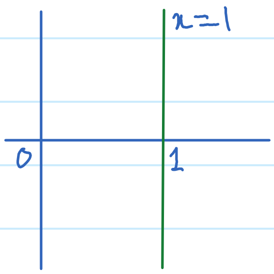
$$z_1 = \sqrt[3]{2}\, e^{i\frac{\pi}{3}}$$

$$z_2 = z_1 e^{i\frac{2\pi}{3}} \quad \text{(rotation of } z_1 \text{ by } \tfrac{2\pi}{3})$$

$$z_3 = z_2 e^{i\frac{2\pi}{3}} \quad \text{(rotation of } z_2 \text{ by } \tfrac{2\pi}{3})$$

Rotation of z_3 by $\frac{2\pi}{3}$ gives us z_1.

Last time, we observed that the image of the line $x = 1$ under the exponential map is a circle. That experiment was done on Mathematica. Now we will explain that result.

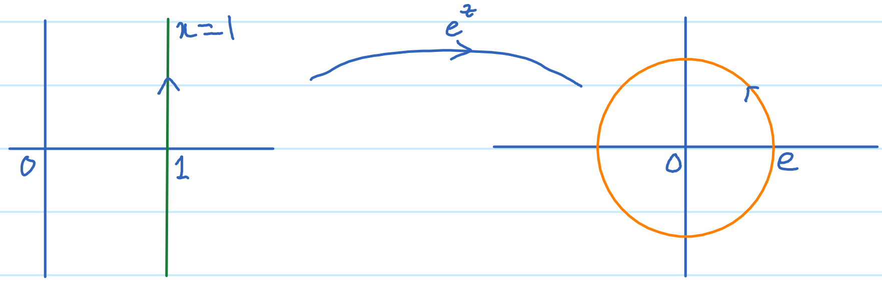


The line $x=1$ has complex parametrization

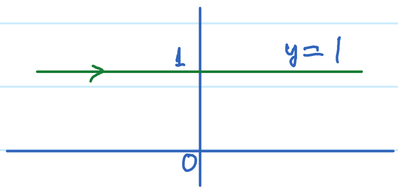
$$z = 1 + it \quad \text{where } t \in \mathbb{R}.$$

The image of point $z = 1+it$ under the exponential map is $e^z = e\, e^{it}$. As t varies, the point

$e\,e^{it}$ has a fixed modulus, which is e, and a varied argument t.
This is why the image of the line $x=1$ is a circle with radius e.



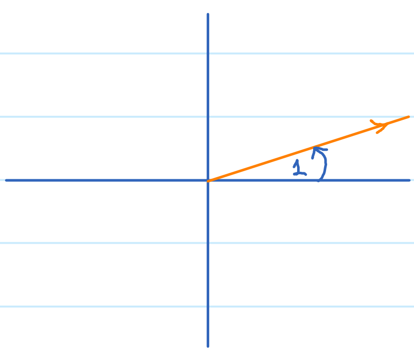
How about the image of the horizontal line $y=1$?



This line has complex parametrization

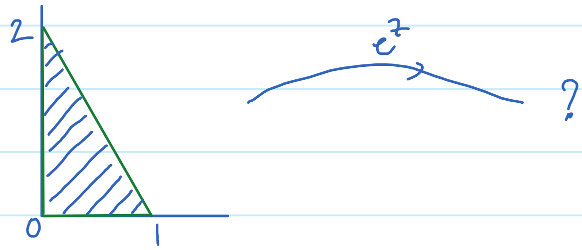
$$z = t + i$$

The image of z is $e^z = e^{t+i} = e^t e^i$.
As t varies, the point $e^t e^i$ has a fixed argument, which is 1, but a varied modulus, which is e^t. As t increases, we travel to the east on the line $y=1$ and our image (under the exponential map) travels toward infinity on the ray $\operatorname{Arg}(z) = 1$.



As t decreases, we travel to the west on the line $y=1$ and our image travels toward the origin.

How to draw the image of a region under the exponential map?
Last time we drew the image of a rectangle under the exponential map.
How about image of a triangle?

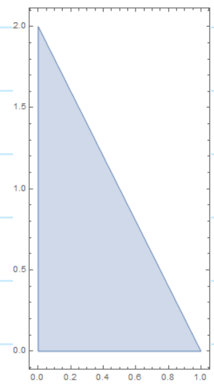


The ParametricPlot command in Mathematica is able to draw a domain (region) given its parametrization. The parametrization of the triangle is

$$\begin{cases} x = t \\ y = s \end{cases}, \quad 0 \le t \le 1, \quad 0 \le s \le 2-2t.$$

In Mathematica, we draw this triangle by

ParametricPlot[{t, s}, {t, 0,1}, {s, 0,2 − 2 ∗ t}]

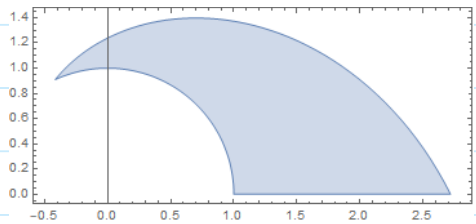


The above command is equivalent to

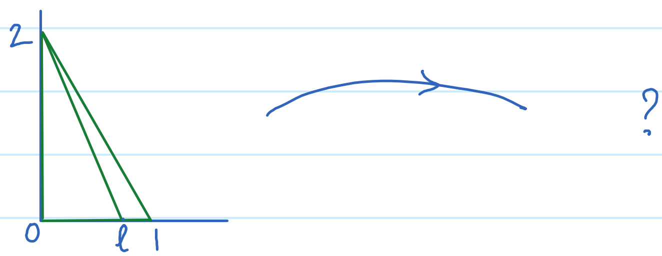
ParametricPlot[$ReIm$[t + I ∗ s], {t, 0,1}, {s, 0,2 − 2 ∗ t}]

To draw the image of the triangle under the exponential map:

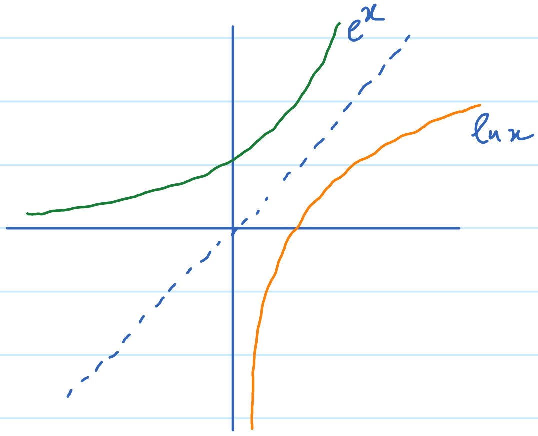
ParametricPlot[$ReIm$[Exp[t + I ∗ s]], {t, 0,1}, {s, 0,2 − 2 ∗ t}]



One can further experiment by drawing a dynamic graph: say, allowing the range of t to be $[0, \ell]$ for $0 < \ell \le 1$. Each ℓ gives a triangle and the image of that triangle under the exponential map. Then let ℓ run from 0.1 to 1.



In Math 251, we defined the logarithm as the inverse function of the exponential function. The exponential function with real variable is an increasing function, so it is one-to-one and has an inverse function.



The exponential function of complex variable, on the other hand, is not one-to-one. In fact, it is periodic. So it doesn't have an inverse function. However, we can ask how to solve the equation $e^w = z$ with unknown w.

Write $w = a + ib$. Then $e^w = e^a e^{ib}$. Write z in polar form $z = re^{i\theta}$. The equation $e^a e^{ib} = re^{i\theta}$ becomes

$$\begin{cases} e^a = r \\ b = \theta + k2\pi \end{cases}$$

Thus,

$$\begin{cases} a = \ln r = \ln|z| \\ b = \theta + k2\pi = \arg z \end{cases}$$

We get $w = \ln|z| + i\underbrace{\arg z}_{\text{multi-valued function}}$

The logarithm of z is defined as

$$\log z = \ln|z| + i\arg z$$

This is a multi-valued function. Strictly speaking, it is not a function in usual sense because one value of z corresponds to infinitely many values of $\log z$.

Ex:

Write $\log(1+i)$ in standard form.

We first write $1+i$ in polar form: $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\, e^{i\frac{\pi}{4}}$.

Thus, $\log(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + k2\pi\right)$ where $k \in \mathbb{Z}$.

To do calculus on the logarithm (taking derivatives, integral, etc), we need to make it a function. The principal logarithm of z is defined as a single-valued version of the logarithm:

$$\operatorname{Log} z = \ln|z| + i\underbrace{\operatorname{Arg} z}_{\text{belongs to } (-\pi,\pi]}$$

In other words, $\operatorname{Log} z$ is a "branch" or "section" of the logarithm of z in which k is chosen such that $\theta + k2\pi \in (-\pi, \pi]$.

Ex:

$$\operatorname{Log}(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + k2\pi\right)$$ where k is chosen such that

$\pi/4 + k2\pi$ belongs to the interval $(-\pi,\pi]$. We see that $k=0$ is the only value of k that satisfies this condition. Therefore,

$$\operatorname{Log}(1+i) = \ln\sqrt{2} + i\frac{\pi}{4}.$$

[See Worksheet for more examples.]