Lecture 7

Monday, April 13, 2020

Recall the definition of the exponential, sine and cosine functions:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$$

or equivalently

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

Sine function:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

or equivalently $$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots$$

Cosine function:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

or equivalently $$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

Other functions can also be defined from these functions, such as

$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{i}\,\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$$

$$\cot z = \frac{\cos z}{\sin z} = i\,\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}$$

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

Let us investigate some geometric properties of the exponential function. The real-variabled function $f(x) = e^x$ can be represented by its graph (a curve). There is not much to say about this graph.

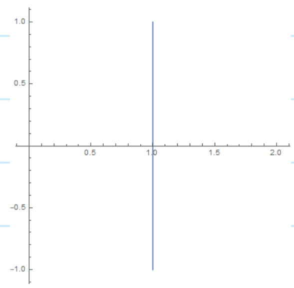
However, the complex-variabled function $f(z)=e^z$ has much richer geometric properties to discuss. The function $f(z)=e^z$ maps each complex number to a complex number. Geometrically, f maps each point on the plane to another point on the plane. Thus, f is a geometric transformation on the plane.

One can ask: what is the image of a line, say $x=1$, under the exponential map? One can use Mathematica to plot the line $x=1$. First, we parametrize the line. There are two ways to write the equation of the line:

* real form: $\begin{cases} x=1 \\ y=t \end{cases}$
* complex form: $z=1+it$.

One can use the command ParametricPlot in Mathematica:

ParametricPlot[{1, t}, {t, −1,1}]

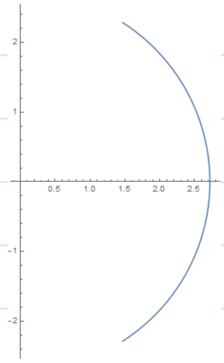


or equivalently

ParametricPlot[ReIm[1 + I ∗ t], {t, −1,1}, AspectRatio → Automatic, AxesOrigin → {0,0}]

To plot the image of this line under the exponential function, we only need to adjust the above command a little bit:

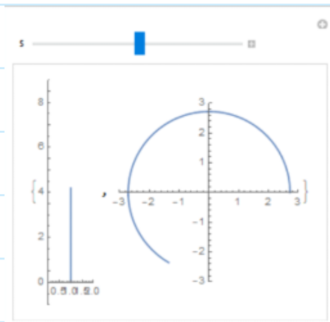
ParametricPlot[ReIm[Exp[1 + I ∗ t]], {t, −1,1}, AspectRatio → Automatic, AxesOrigin → {0,0}]



One can create a dynamic plot to better visualize the image of the line. (See the instruction file Mapping properties of the exponential function).

line

$z = 1+it$



curve $e^z = e^{1+it}$

$= e\cos t + i e \sin t$

We can observe from the picture that as t travels from 0 to 2π ($\sim 6.28\ldots$) the image of the line is a circle. We can also see that when t reaches 2π, we return to the original point. This hints a periodic behavior of the exponential function. Consider the equation

$$e^z = e^w.$$

Write $z = a+ib$ and $w = c+id$. Then

$$e^z = e^a e^{ib} \rightsquigarrow \text{modulus is } e^a, \text{ argument is } b$$

$$e^w = e^c e^{id} \rightsquigarrow \text{modulus is } e^c, \text{ argument is } d$$

Thus,

$$\begin{cases} e^a = e^c \\ b = d + k2\pi \end{cases}$$

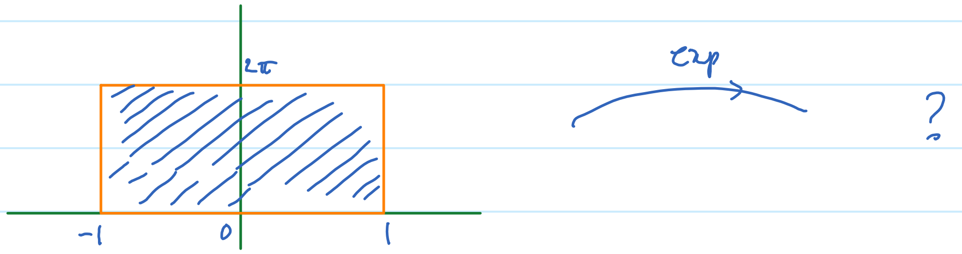
In other words, $a = c$ and $b = d + k2\pi$

We conclude that

$$e^z = e^w \text{ if and only if } z = w + ik2\pi \text{ for some } k \in \mathbb{Z}.$$

This means e^z is a periodic function with period 2π.

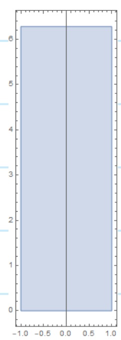
We have seen how to sketch the image of a curve under the exponential map. Now let us sketch the image of a region, say the rectangle $[-1,1] \times [0, 2\pi]$.



The rectangle can be parametrize as $\begin{cases} x = t, & -1 \le t \le 1 \\ y = s, & 0 \le s \le 2\pi \end{cases}$

We can draw this rectangle on Mathematica using the Command Parametric plot.

ParametricPlot[ReIm[t + I * s], {t, −1,1}, {s, 0,2 * Pi}]



To draw the image of this rectangle, we only need to adjust the previous command a little bit:

ParametricPlot[ReIm[Exp[t + s * I]], {t, −1,1}, {s, 0,2 * Pi}, PlotRange → Full]